186203

RUDIMENTARY TREATISE

ON

ARITHMETIC,

WITH

FULL EXPLANATIONS OF ITS THEORETICAL PRINCIPLES,

AND

NUMEROUS EXAMPLES FOR PRACTICE.

For the Use of Schools,

AND FOR SELF-INSTRUCTION.

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SECOND EDITION, CORRECTED.

"The rules of Arithmetic are formed generally for the use of those who have not arrived at an age when the reflective and reasoning faculties are sufficiently exercised and strengthened to enable them to understand fully the principles of the rules which they follow; but it may be justly doubted whether the acquiescence in this mode of education is not much too general, and whether habits of investigation and inquiry are not checked, at least, if not destroyed, by teaching the student to follow merely mechanical rules, in which the understanding takes no part."—*Professor Peacock: Encyclopædia Metropolitana, art. Arithmetic.*

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PREFACE.

In this little book I have endeavoured to expound, upon simple and rational principles, the rudiments of the SCIENCE OF ARITHMETIC. With *rules* I have given *reasons*: and although the work is designed chiefly for schoolboys and young persons, yet, contrary to the usual practice, I have chosen to regard the learner less as an arithmetical machine than as an intellectual being. I venture to hope, that what I have here done may meet with some degree of countenance from Schoolmasters and Teachers; and that it may also prove acceptable to the solitary and self-dependent student. This is not an unreasonable hope: for, although so-called Treatises on Arithmetic are very numerous, the number of books really deserving of the appellation is but few. As I have reserved no room here for even the most summary analysis of the following pages, I must leave them to the candid examination of those who may be interested in the progress of this class of educational books. I trust no fault will be found with me for the familiar and colloquial form of exposition I have adopted: any attempt at elevation of style, in works of this kind, is wholly misplaced. I have imagined my own pupils before me; and I have addressed them as I was formerly in the habit of doing in oral instruction. My sole aim has been to be intelligible, and to invest the subject with what interest I could consistently with the preservation of scientific accuracy: but if there be one part more than another, to which I would invite special attention, it is the subject of *Decimals*, towards the end of the book.

J. R. YOUNG.

LONDON, 1853.

*** A Key to the work is now published: besides solutions in full to all the Exercises, it will furnish some additional instructions for the otherwise unassisted learner.

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ARITHMETIC.

(1.) The marks used in Arithmetic are

1, 2, 3, 4, 5, 6, 7, 8, 9;

which stand for *one*, *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine.**

These marks are called *figures;* and by help of these figures, and another mark, 0, to stand for *nought*, or *nothing*, any number may be written down.

The mark 0 or *nought*, may also be called a *figure;* so that there are ten different marks or *figures* used in Arithmetic.

(2.) You must remember that a *figure* is only *one* of these marks: when you see two or three or more of them written side by side, you see a *number:* thus, 24 is a *number*, of two *figures;* it stands for *twenty-four;* also 37 is a *number* of two *figures;* it stands for *thirty-seven:* and so on. Twenty-four means *two tens* and *four; thirty-seven* means *three tens* and *seven.* And in like manner *forty-eight* means *four tens* and *eight;* and this number, written in figures, is 48.

You see then, that, in a number of two figures, the first, or left-hand figure, tells us how many *tens* there are in the number; and the second, or right-hand figure, tells us how many *ones*, or *units* there are in the number, besides the tens: *one*, you are to remember, is also called *unit*, or *unity.*

(3.) From what has now been said, you see that the word *number* does not mean the same thing as the word *figure:* there are only ten different figures, or single marks, but by joining two or more of these together, we may write down as many *numbers* as we please. The single figures themselves are also called *numbers*, as well as sets of two, three, or more figures: thus, 5, 7, 6, &c., are numbers of one figure each; 57, 75, 76, &c., are numbers of two figures; and 576, 756, &c., are numbers of three figures. The number 57 is *fifty-seven*, the number 75 is *seventy-five*, the number 76 is *seventy-*

* Arithmetic may be defined as the science which teaches how to perform computations by *numbers.* It would perhaps be of but little use to a beginner to give a formal definition of Arithmetic in the text.

six, and so on. The number 576 is *five hundred and seventy-six*, the number 756 is *seven hundred and fifty six;* and in any number of *three* figures, the first figure on the *left hand* tells us how many *hundreds* there are in the number; the next figure tells us how many *tens* there are, besides the hundreds; and the last figure tells us how many *units* there are, besides the hundreds and tens: there may be *no units* after the hundreds and tens; if so, a *nought* or 0, is put for the last figure: thus, *five hundred and seventy*, would be written in this way, 570; and *seven hundred and fifty*, would be written 750. Should there be no *tens* after the hundreds, then, in the same way, a 0 is put *in the place of tens:* thus, 506, means five hundred, no tens, and six units; that is, *five hundred and six;* also, 605 means *six hundred and five;* and 600 means six hundred, without any tens or units besides. You see, therefore, that if you write down a single figure, you mean so many units; but if you put a *nought* to the right of it, you mean ten times as many, and if two *noughts*, one hundred times as many.

(4.) When a number has four figures, as, for instance, the number 3562, the first figure on the left hand tells us how many *thousands* there are in the number; the next figure, how many *hundreds* besides; the third figure, how many *tens;* and the fourth, or last figure, how many *units* besides; so that the number just written is *three thousand five hundred and sixty-two:* if, instead of *five* hundreds, there had been *no* hundreds, the number would have been written 3062; that is, *three thousand and sixty-two.* In like manner, 3502 is *three thousand five hundred and two;* also, 3002, is *three thousand and two;* and 3000, is *three thousand* only.

(5.) From the explanations which have now been given, you see that when a figure stands by itself, that is, without any figures beside it, it stands for a certain number of *units:* thus, 6 stands for *six units*, or *six ones;* and that when a figure does not stand by itself, but at the *right-hand end* of a row of figures, it still stands for *units:* thus, the 6 in the number 346 is still six *units;* but the 6 in the number 562 stands for ten times 6, or sixty; the 6 in the number 4637 stands for ten times sixty, or six hundred; the 6 in the number 6253 stands for ten times six hundred, or six thousand. It is on this account that we say that the last figure of any number occupies the place of *units;* the next figure,

on the left, the place of *tens;* the next, the place of *hundreds;* the next, the place of *thousands;* and so on, as in the following Table, which is called the

Numeration Table.

3	7	5	2	6	8	4	3	6	2	9	7
&c. hundreds of thousands of millions.	tens of thousands of millions.	thousands of millions.	hundreds of millions.	tens of millions.	millions.	hundreds of thousands.	tens of thousands.	thousands.	hundreds.	tens.	units.

(6.) The number written above, which is a number of twelve figures, is therefore three hundred and seventy-five thousand two hundred and sixty-eight *millions*, four hundred and thirty-six *thousand*, two hundred and ninety-seven. If we were to put another figure before the first figure 3 above, we should have a number of thirteen figures, and the new figure would be in the place of *billions;* another new figure put before this would be in the place of *tens of billions*, and so on. You should *learn* this table, by repeating the words, units, tens, hundreds, thousands, &c., till you can say them in order without looking at the book; and you must notice, that whichever of these you pronounce, the next you pronounce is always *ten* times *it:* thus, *ten* is ten times *unit;* a *hundred* is ten times *ten;* a *thousand* is ten times a *hundred*, and so on.

(7.) Until you have learned the Numeration Table, you will not know the meaning of a large number written in figures. Suppose the number 465287 were shown to you; you could tell what it means only by knowing the Numeration Table. You would point to the 7 and say *units*, to the 8 and say *tens*, to the 2 and say *hundreds*, to the 5 and say *thousands*, to the 6 and say *tens of thousands*, and to the 4 and say *hundreds of thousands;* and you would thus know

the number to be four hundred and sixty-five *thousand*, two hundred and eighty-seven. And in this way you are now required to write in *words* the following numbers: *—

Exercises in Numeration.

1.	2763	35162	45280
2.	56106	82030	910257
3.	173004	6789523	3486025
4.	1142060	1110111	4362800
5.	64370253	99874062	35006200
6.	73892531	875062035	107926500
7.	7539336210	326972573971	415862314203
8.	730254062810	173004202604.	502130065080

(8.) Write the following numbers in figures:—

1. A mile contains one thousand seven hundred and sixty yards.
2. It has been found that there are more than five hundred and forty-six thousand persons in the world who are deaf and dumb.
3. The expense of building London Bridge was two millions of pounds.
4. The expense of the Britannia Tube, over the Menai Straits, was six hundred and twenty-one thousand eight hundred and sixty-five pounds.
5. The quantity of gold collected at California, in the year 1850, is estimated at three hundred and twelve thousand five hundred ounces.
6. The money received by the London and North-western Railway for passengers and goods, during the first half of the year 1851, was one million one hundred and seventy-seven thousand five hundred and eighty pounds.
7. The number of visitors to the Great Exhibition, on Tuesday, the 7th of October, 1851, was one hundred and nine thousand nine hundred and fifteen. [This was the greatest number on any one day.]
8. The money received by the Great Exhibition amounted altogether to five hundred and five thousand one hundred and seven pounds.
9. In the month of September, 1851, there were one million one hundred and fifty-five thousand two hundred and forty visits paid to the Exhibition.

* The answers to all the questions and exercises are placed at the end.

10. The greatest number of visits to the Exhibition in any one month was in July; the number in that month was one million three hundred and fourteen thousand one hundred and seventy-six.
11. The total number of visits to the Exhibition, without counting the closing day and certain private days, was six million seven thousand nine hundred and forty-four.*
12. The number of people in Great Britain and the British Islands in 1851 was twenty million nine hundred and thirty-six thousand four hundred and sixty-eight.

(9.) From these exercises you will see that the ten figures of arithmetic are sufficient for the purpose of expressing any number, however great; and that the reason why so few are enough, is, that each figure changes its meaning as its *place* in a number is changed. The value of a figure in any particular *place* is called its *local* value; thus, the *local value* of the 6 in 263 is sixty; and the *local value* of the 2 is two hundred: the local value being always *ten times* what it would be if the figure were in the next place on the right. This ten-fold increase in the local value of a figure, when it is advanced one place from right to left, is the reason why our system of *notation* in arithmetic is called the *decimal system*.†

It may be proper to mention here, that the figures 1, 2, 3, &c. are also called *digits*; and that 0, or *nought*, is also called *cipher*, or *zero*.

(10.) It may also be further noticed, that when we have to express a very large number in words, it is convenient to separate the figures of it into *periods* of three figures each, by putting a comma before the *last three* figures, another comma before the next three, and so on: thus, the large number by which I have illustrated the Numeration Table, when the figures are divided into periods, is 375,268,436,297. The advantage of this is, that as the leading figure of each period occupies the place of *hundreds*,—that is, of hundreds simply, or hundreds of thousands, or hundreds of millions, &c., the number is more readily put into words. In the tables published by order of Government, relating to revenue, population, &c., this plan is always adopted.

ADDITION.

(11.) The rule for *adding* a set of numbers together, so as to find the *sum* of them all, is called the rule for Addition; it is the first rule in Arithmetic, and is as follows:—

* Including the six exceptional days; namely, the opening day, the two days at one pound, the two exhibitors' days, and the closing day, the number in the 144 days was about 6,170,000.

† The learner may be here informed, that the particular marks or symbols used in any science,—as, for instance, the *notes* in music, and the characters or symbols 1, 2, 3, &c. in arithmetic,—constitute the *notation* of that science.

RULE 1. Place the numbers to be added under one another, so that the *units* may all be in the first column on the *right*, the *tens* in the second column, the *hundreds* in the third column, and so on.

2. Add up the column of *units;* that is, find the *sum* of the units in this column; if this sum be a number of only one figure, put this figure down under the unit column; but if it be a number of more than one figure, it is *not* to be put down; the *last* figure of it only, that is, the figure in the *units' place*, is to be put down, and the number expressed by the remaining figure or figures, after the one put down is rubbed out, is to be *carried* to the column of *tens*, and added up with that column. In like manner, put down the *last* figure of the number which is the *sum* of this second column, and *carry* the number expressed by the remaining figures, when the one put down is rubbed out, to the next column, or column of hundreds, and so on till you reach the last column, the *whole* amount of which is to be put down.

For example: Suppose we have to add together the numbers 327, 241, and 58. Then, writing the numbers under one another, so that the *units* may all be in the first column on the right, the *tens* in the second column, and the *hundreds* in the third, as in the margin, we should proceed as follows:—

$$\begin{array}{r} 327 \\ 241 \\ 58 \\ \hline 626 \\ \hline \end{array}$$

8 and 1 are 9, and 7 are 16; therefore there are sixteen units in the first column, and as this number consists of *two* figures, 1 and 6, we put down only the 6, and carry the 1 to the next column, and say 1 and 5 are 6, and 4 are 10, and 2 are 12; we therefore put down 2 and carry 1 to the next column, saying 1 and 2 are 3, and 3 are 6, which we put down, and thus find the sum of the numbers to be 626; that is, six hundred and twenty-six.

Again: Suppose we have to add together the numbers 7625, 3253, 1802, and 211. Writing the numbers under one another, as before, we say, 1 and 2 are 3, and 3 are 6, and 5 are 11, 1 and carry 1; 1 and 1 are 2, and 5 are 7, and 2 are 9; this being a *single figure*, we put it down, and have nothing to carry to the next column; 2 and 8 are 10, and 2 are 12, and 6 are 18, 8 and carry 1; 1 and 1 are 2, and 3 are 5, and 7 are 12; and having now reached the last column, the whole amount 12 is put down;

$$\begin{array}{r} 7625 \\ 3253 \\ 1802 \\ 211 \\ \hline 12891 \\ \hline \end{array}$$

so that the sum of the numbers is 12891; that is, twelve thousand eight hundred and ninety-one.

We shall go through the work of but one other example: Add together the numbers 57632, 804, 70300, 731, and 33. Writing the numbers in columns, as before, we say, 3 and 1 are 4, and 4 are 8, and 2 are 10, 0 and carry 1; 1 and 3 are 4, and 3 are 7, and 3 are 10, 0 and carry 1; 1 and 7 are 8, and 3 are 11, and 8 are 19, and 6 are 25, 5 and carry 2; 2 and 7 are 9; 7 and 5 are 12. So that the sum is, one hundred and twenty-nine thousand five hundred.

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 57632
   804
 70300
   731
    33
------
129500
------
```

(12.) You will easily see that the work of these examples is right: thus, looking back to the first example, you see that the column of units amounts to 16 units; that is, to one *ten* and six *units* over; the one *ten* is carried, as it ought to be, to the column of *tens*, and only the six units put in the units' place. The *tens* amount to 12 tens; that is, to ten tens, or *one* hundred, and two tens over; these two tens are therefore put in the tens' place, and the *one* hundred carried to the column of hundreds; the sum of this column is found to be *six;* so that we have got the right number of *hundreds*, the right number of *tens*, and the right number of *units*, in the whole. And in the same way you may convince yourself that each of the other examples is correct, and that the rule must lead you to the true sum or amount in all cases.

(13.) There is a way of *proving* whether the columns are added up without error, given in most books on arithmetic; but it would cause you more trouble than to do the work over again; you had better therefore make up your mind to perform the addition a second time, when you are not quite sure that there is no mistake: this *second* time you should commence at the *top* of each column, and add *downwards;* thus, if you wish to try the correctness of example 2, above, begin at the *top* of the units' column, and say, 5 and 3 are 8, and 2 are 10, and 1 are 11, 1 and carry 1; 1 and 2 are 3, and 5 are 8, and 1 are 9; and so on.

Exercises in Addition.

1. Add together the numbers 342, 165, and 34.
2. Find the sum of 87, 273, and 49.
3. Find the sum of 2860, 1723, 41, and 17.
4. Add together 5693, 482, 6297, and 13.
5. Add together 17341, 9203, 510, and 20061.
6. What is the sum of 35208, 62, 187, and 762070?
7. Add up 7407003, 169205, 4853, 79, and 382.

8. Add the following numbers together; namely, two thousand and four, seven thousand and thirty-five, one hundred and one thousand and nine, seventeen thousand and forty-eight, and two hundred and one.

9. On the London and Birmingham Railway, the Primrose Hill tunnel is one thousand one hundred and twenty yards in length; the Weedon tunnel, four hundred and eighteen yards; the Kilsby tunnel, two thousand three hundred and ninety-eight yards. On the Great Western Railway, the Box tunnel is three thousand two hundred and twenty-seven yards. On the Manchester and Leeds Railway, the Littleborough tunnel is two thousand eight hundred and sixty-nine yards; and the Merstham tunnel, on the London and Brighton Railway, is one thousand seven hundred and eighty yards. What is the sum of the lengths of these six tunnels?

10. By order of Government, all the people in this kingdom are counted every ten years; this counting is called taking a *census* of the population. In 1841 and 1851 the numbers were found to be as follow:—

	Census of 1841.	*Census of* 1851.
In England and Wales...	15911725	17922768
Scotland	2628957	2870784
Islands in British Seas	124079	142916

What was the amount of the population in 1841 and 1851?

11. The number of persons visiting the Great Exhibition of 1851, during the last week, was as follows; namely,

On Monday,	Oct. 6	107815
Tuesday,	Oct. 7	109915
Wednesday,	Oct. 8	109760
Thursday,	Oct. 9	90813
Friday,	Oct. 10	46913
Saturday,	Oct. 11	53061

How many visits were made during the last week altogether?

12. Find how many days the Exhibition was open, and how many visits were paid to it altogether, from the following statement:—

	Number of Days.	*Number of Persons.*
May	27	734814
June	25	1133114
July	27	1314176

August 26	....	1023435
September 26	....	1155240
October 13	....	808237

13. The number of persons who *emigrated* from this kingdom,—that is, who left it to live in other countries,—in the years 1849 and 1850, was as follows; namely,

	In 1849.	*In* 1850.
From England	212124 persons;	214612 persons
Scotland	17127 ,,	15154 ,,
Ireland	70247 ,,	51083 ,,

What was the amount of emigration during these two years?

14. In the year ending on the 5th of January, 1851, there were 159 London newspapers, in which there appeared 891650 advertisements, and 222 English country newspapers, in which 875631 advertisements appeared. There were also 110 Scotch newspapers, with 249141 advertisements; and 102 Irish papers, with 236128 advertisements. Find the total number of newspapers, and the total number of advertisements.

(14.) It is proper that you should now be told, that besides the marks or symbols used in arithmetic for *numbers*, some other marks are also used, instead of common words, for the *operations* of arithmetic. The operation you have just performed, in each of the foregoing examples, is the operation of *addition.* Now there is a mark or sign used for addition; it is an upright cross, thus +, placed between the figures or numbers to be added, and it is called *the sign of addition*, and is read *plus;* there is also a sign for the *result* of any operation, as for instance, for the *sum* or total of a set of numbers; it is called *the sign of equality*, and is written thus = : whenever you see this sign, you are to understand that the words *equal to* are meant by it. By using these two signs, you may write down any example in addition, with the answer or result put against it, without employing any *words:* thus, taking the first example at page 6, you might write it so:—

$$327 + 241 + 58 = 626;$$

and if you were asked to *read* this, you would say 327 *plus* 241 *plus* 58, are *equal to* 626, or *equals* 626.

In like manner, the second example, with the answer or result, would stand thus:—

$$7625 + 3253 + 1802 + 211 = 12891;$$

and you yourself can now put the third, or any other example, in the same form, and can read it when you have done so. You must really attend to these signs of addition and of equality, because they will be often used hereafter.

SUBTRACTION.

(15.) WE now come to the second rule of arithmetic, which is called the rule of *Subtraction:* it teaches us how to take, that is, to *subtract*, the smaller of two numbers from the greater, and so to find the difference or *remainder*. If both numbers are numbers of *single figures* only, you do not feel the want of a particular *rule:* if you are asked to find the difference between the number 8 and the number 5, that is, if you are asked to *subtract* 5 from 8, you can easily see that the difference or remainder must be 3; and you also know, that if 2 be subtracted from, that is, taken away from, 9, the remainder will be 7; that if 4 be subtracted from 6, the remainder will be 2; and so on: but if each be a number of *several* figures, you will want a rule to guide you to the correct remainder: this rule is as follows:—

RULE 1. Write the smaller number under the greater, taking care, as in addition, to place units under units, tens under tens, hundreds under hundreds, &c.

2. Then, beginning with the units, as in addition, subtract the lower figure from the upper: this you can easily do if the upper figure be the *greater* of the two; but if it be the *less*, you must fancy 10 added to it, which is sure to make it greater, so that you can now subtract the lower figure, and put the remainder underneath.

3. If you have increased the upper figure by 10, you must *carry* 1 to the *next* figure to be subtracted; if this next figure, thus increased by 1, be greater than the figure above it, 10 must be added to the upper figure, as before; and after the remainder is put down, 1 must be carried to the next figure; and so on.

You see, therefore, that the method is to subtract the *units* of the lower or smaller number from the *units* of the upper, the *tens* from the *tens*, the *hundreds* from the *hundreds*, and so on, always adding 10 to the upper figure *when it is less than the figure under it*, and taking care, in such a case, to carry 1 to the next figure: it is only when 10 is thus added

to an upper figure that 1 is to be carried to the next lower one. The following examples will explain the operation:—

1. Subtract 625 from 6879.

Placing the smaller number under the greater, as in the margin, we say, 5 from 9, and 4 remain; 2 from 7, and 5 remain; 6 from 8, and 2 remain; and as nothing is taken from the upper figure 6, the complete remainder is 6254, or 6 thousand 2 hundred and 54.

6879	greater
625	less
6254	remainder

As in this example each of the lower figures is less than the figure above it, the subtraction is performed without adding 10 to any upper figure: in the next example such is not the case.

2. Subtract 13758 from 23596.

The numbers being written as before, we see that the figure 6 in the units' place of the upper number is less than the figure 8 below it; we therefore fancy 10 to be added to the 6, making it 16, and say, 8 from 16, and 8 remain, carry 1; 6 from 9, and 3 remain; 7 from 15, and 8 remain, carry 1; 4 from 13, and 9 remain, carry 1; 2 from 2, and nothing remains; therefore the remainder is 9838.

23596
13758
9838

3. Subtract 3506285 from 72311075.

Instead of repeating the word *remain* at every subtraction, it is better to proceed as follows: 5 from 5, nought; 8 from 17, 9, carry 1; 3 from 10, 7, carry 1; 7 from 11, 4, carry 1; 1 from 1, nought; 5 from 13, 8, carry 1; 4 from 12, 8, carry 1; 1 from 7, 6.

72311075
3506285
68804790

(16.) The truth of this rule for subtraction may be shown in a few words:—Adding 10 to any *figure* of a number is the same as adding 1 to the figure before it: thus, the number 75, is 7 *tens* and 5; if I add 10 to the 5, I make it 7 tens and 1 ten and 5, that is 85. Again; the number 623, is 6 hundreds, 2 tens, and 3; if I add 10 to the 2, I make it 6 hundreds, 12 tens, and 3; or 6 hundreds, 10 tens (which is another hundred), 2 tens, and 3, that is 723, and so in other cases; so that adding 10 to a figure is, in fact, adding 1 to the figure before it. The rule tells us, that whenever we add 10 to an upper figure, we must subtract an additional 1 from the figure before it; therefore, the 1 that has been added to an upper figure, for convenience, is immediately afterwards taken away, so that all is brought right again.

I shall add two or three examples more, with the remainders put down for you to look over; and shall then give some exercises in subtraction for you to find the remainders yourself.

6803029	34510381	8057130600
2516017	6232045	148112354
4287012	28278336	7909018246

(17.) You can *prove* whether the subtraction is correctly performed, by *adding* the remainder to the number which has been subtracted; the *sum* ought to be the top number: thus, taking the first of the three examples just given, you would say, 2 and 7 are 9; 1 and 1 are 2; 0 and 0 are 0; 7 and 6 are 13, 3 and carry 1; 1 and 8 are 9, and 1 are 10, 0 and carry 1; 1 and 2 are 3, and 5 are 8; 4 and 2 are 6. And as the figures thus obtained are those of the top number, we conclude that the work is right.

Exercises in Subtraction.

1. Subtract 375 from 846, and 1237 from 2865.
2. What is the difference between 36207 and 72098?
3. Take 7992 from 18097, and 300043 from 1001251.
4. Subtract seven thousand and fifty-three, from a hundred and eleven thousand and two.
5. Subtract thirteen thousand one hundred and seventeen, from twenty-two thousand and five.
6. What is the difference between one million three hundred and two thousand and forty-two, and three million one hundred and eleven?
7. The three greatest generals in modern times—the Duke of Wellington, Napoleon Bonaparte, and Marshal Soult—were all born in the same year, 1769; the last died in November, 1851: how old was the Duke of Wellington then?
8. The population of Ireland in the year 1841 was 8175124, and in 1851 it was 6515794: find by how many people the population had decreased in these ten years.*
9. The population of Ireland in the year 1821 was 6801827, and in the year 1831 it was 7767401: what was the increase in these ten years?
10. The population of Great Britain and its adjacent Islands

* The learner should be required to state the numbers in these exercises in *words;* and to write his results both in figures and words: he may divide the figures into *periods*, as explained at page 5.

in the year 1841 was 18664761, and in 1851 it was 20936468: find the increase in these ten years.

11. The number of Season Tickets for the Great Exhibition, sold before the building was opened, was nineteen thousand five hundred and seven; of these, eight thousand six hundred and fifteen were Ladies' Tickets: how many of them were Gentlemen's Tickets?

12. The number of visits paid to the British Museum in the year 1850 was 1098863, and to Hampton Court Palace 221119: how many visits were paid to the former place more than to the latter?

13. The Gross Revenue of the Post Office* for the year ending on the 5th of January, 1851, was 2264684 pounds, and the cost of management was 1460785 pounds: what was the Net Revenue for the year?

14. The total number of passengers conveyed on the Railways of the United Kingdom in the half-year ending on the 30th of June, 1850, was 31766503; and in the half-year ending on the 31st of December, 1850, the total number was 41087919: what was the increase in the number of passengers in the last half-year?

15. The gross receipts of the London and Brighton Railway during the week ending Nov. 22, 1851, were, for Passengers, 6217 pounds; for Goods, 2135 pounds. The gross receipts for Passengers and Goods, during the corresponding week of the preceding year, were 8149 pounds: find how much the receipts had increased.

16. The population of Great Britain and the neighbouring Islands in 1851 was 20936468; the population of England and Wales alone was 17922768, and the population of the British Islands alone was 142916: what was the population of Scotland?

17. The salaries paid to the officers employed by the Custom-House in 1849 were as follows: salaries in England, 550236 pounds; salaries in Scotland, 62115 pounds; salaries in Ireland, 57903 pounds. The amount of Custom-House duty, collected in that year, was 22481339 pounds: what was the net amount received after these salaries were paid?

* By *revenue* of the Post Office is meant *income* of the Post Office; and *gross* revenue, or *gross* receipts, means the money received before the expenses of management are subtracted; when these expenses are taken from the *gross* income, the remainder is called the *net* income.—See Exercise 17.

18. In the year 1849 there were 578159 children born in England and Wales; of these 295158 were males. In the same year 440853 persons died; of these 221801 were males; you are required to find how many females were born in 1849, and how many died.
19. What is the difference between 365 + 2041 + 109, and 7530 + 1623 + 87 + 3406?
20. What is the difference between 112104 + 3820 + 3268, and 2389 + 103403 + 13400?
21. Find the difference between 462873 + 5962 + 304 + 19871, and 1735 + 902603 + 72 + 139.

(18.) The last three exercises bring into use the *sign of addition*, explained at page 9. There is also a *sign of subtraction*, which it is equally necessary that you should remember; it is the little mark —. This sign, placed before a number, means that the number is to be *subtracted.* By using this sign, which is called *minus*, we may express an example in subtraction without words, the sign of equality, =, being placed before the remainder: thus, the first example, page 11, may be written 6879—625 = 6254; the second example may be written 23596—13758 = 9838; the third may be written 72311075—3506285 = 68804790; and so of the other examples. If you were asked to *read* the first of these, you would say, 6879 *minus* 625, *equals* 6254: you can from this read the others without any help; and I dare say you could even read the following, namely,

$$24 + 36 - 17 - 41 + 13 - 11 - 2 = 2;$$

but in case you should be puzzled, I will read it for you: it is 24 *plus* 36 *minus* 17 *minus* 41 *plus* 13 *minus* 11 *minus* 2, *equals* 2; the meaning of which is, that if from the *sum* of 24, 36, and 13, the *sum* of 17, 41, 11, and 2, be *subtracted*, the remainder will be 2.

MULTIPLICATION.

(19.) We now come to the third rule in Arithmetic,—the rule for *multiplication*,—which teaches us how to find the sum of a set of *equal* numbers without our taking the trouble to put them all down, and add them together, as in addition.

If we have to find the sum of *two* equal numbers, we put down only *one* of those numbers, and multiply it by 2, according to the rule to be given presently; if we have to find the sum of *three* equal numbers, we put down one of them, and multiply by 3; and in like manner if we have to find the sum of *eight*, or *nine*, &c., equal numbers, we multiply one of them by 8, or 9, &c. In this manner we discover the sum required very soon.

The number we multiply another number by is called the *multiplier;* and the other number the *multiplicand:* the result of the operation, and which in addition would be called the *sum*, is here called the *product.*

Whatever be the multiplicand, and whatever be the multiplier, the operation could be described in a single rule; but it will be easier for you if I divide the general rule into two particular rules: I shall therefore do this; but before you can use either rule, you must learn the *Multiplication Table*, which I here give. This table you must repeat in this way: —Twice 1 are 2; twice 2 are 4; twice 3 are 6, &c. Three times 1 are 3; three times 2 are 6; three times 3 are 9, &c. Four times 1 are 4; four times 2 are 8; four times 3 are 12, &c. &c. When you say *twice* any number, you are said to *multiply* that number by 2; when you say 3 times, you are said to *multiply* by 3, and so on; and the number that results is the *product:* thus, when you say 4 times 6 are 24, the *multiplier* is 4, the *multiplicand* is 6, and the *product* is 24. You must remember this. When you say 4 times 8 are 32, if you were asked what is the *multiplier*, what is the *multiplicand*, and what is the *product*,—what would you answer? *

* The learner may, if he please, commit to memory, at first, only a part of the table on the next page, and may select from the exercises that follow, such of them as require only those multipliers within the range of his knowledge of the table. After some practice in these, another portion of the table may be learnt. Simple as the Multiplication Table appears to the arithmetician, it should be regarded by every teacher as a thing of no small labour and difficulty to a mere beginner.

MULTIPLICATION TABLE.

Twice		3 times		4 times		5 times		6 times		7 times		8 times		9 times		10 times		11 times		12 times	
1 are	2	1 are	3	1 are	4	1 are	5	1 are	6	1 are	7	1 are	8	1 are	9	1 are	10	1 are	11	1 are	12
2	4	2	6	2	8	2	10	2	12	2	14	2	16	2	18	2	20	2	22	2	24
3	6	3	9	3	12	3	15	3	18	3	21	3	24	3	27	3	30	3	33	3	36
4	8	4	12	4	16	4	20	4	24	4	28	4	32	4	36	4	40	4	44	4	48
5	10	5	15	5	20	5	25	5	30	5	35	5	40	5	45	5	50	5	55	5	60
6	12	6	18	6	24	6	30	6	36	6	42	6	48	6	54	6	60	6	66	6	72
7	14	7	21	7	28	7	35	7	42	7	49	7	56	7	63	7	70	7	77	7	84
8	16	8	24	8	32	8	40	8	48	8	56	8	64	8	72	8	80	8	88	8	96
9	18	9	27	9	36	9	45	9	54	9	63	9	72	9	81	9	90	9	99	9	108
10	20	10	30	10	40	10	50	10	60	10	70	10	80	10	90	10	100	10	110	10	120
11	22	11	33	11	44	11	55	11	66	11	77	11	88	11	99	11	110	11	121	11	132
12	24	12	36	12	48	12	60	12	72	12	84	12	96	12	108	12	120	12	132	12	144

(20.) You can easily satisfy yourself of the *truth*, and also of the *use* of this table, by taking out of it any multiplicand, and any multiplier,—by writing down the multiplicand as often as there are *units* in the multiplier, and then adding all these *equal* numbers together. You will find that the *sum* of them is always equal to the *product* put down in the table. Thus, the table tells us that 6 times 7 are 42; and we find, by addition, that *six* 7's *are* 42: thus,

$$7 + 7 + 7 + 7 + 7 + 7 = 42.$$

In like manner, we are told by the table that 5 times 8 are 40; and we know, by addition, that

$$8 + 8 + 8 + 8 + 8 = 40;$$

and so in every other case in which the equal numbers are each not greater than 12, and not more than twelve of them are to be added together. When any multiplicand *is* greater than 12, and the multiplier *not* greater than 12, the table will still help us to the product by aid of the first of the two rules I promised; which is as follows:—

1. *When the Multiplier is not greater than* 12.

RULE 1. Place the multiplier under the multiplicand, *units* under *units*.

2. Then, commencing at the units-figure of the multiplicand, multiply each figure, in succession, by the multiplier, and put the product *under* that figure, taking notice, however, that whenever any of these products is a number of *two* or *three figures*, the *right-hand figure only* is to be put down, and *the rest carried to the next product*, as in addition.

Ex. 1. Multiply 2683 by 2.

Having placed the 2 under the 3, as in the margin, I say, twice 3 are 6; twice 8 are 16, 6 and carry 1; twice 6 are 12, and 1 carried are 13, 3 and carry 1; twice 2 are 4, and 1 carried are 5. Therefore the product is 5366.

$$\begin{array}{r} 2683 \\ 2 \\ \hline 5366 \\ \hline \end{array}$$

2. Multiply 728365 by 3.

3 times 5 are 15, 5 and carry 1; 3 times 6 are 18, and 1 are 19, 9 and carry 1; 3 times 3 are 9, and 1 are 10, 0 and carry 1; 3 times 8 are 24, and 1 are 25, 5 and carry 2; 3 times 2 are 6, and 2 are 8; 3 times 7 are 21. Therefore the product is 2185095.

$$\begin{array}{r} 728365 \\ 3 \\ \hline 2185095 \\ \hline \end{array}$$

3. Multiply 276023 by 8.

8 times 3 are 24, 4 and carry 2; 8 times 2 are 16, and 2 are 18, 8 and carry 1; 8 times 0 are 0, and 1 are 1; 8 times 6 are 48, 8 and carry 4; 8 times 7 are 56, and 4 are 60, 0 and carry 6; 8 times 2 are 16, and 6 are 22. Therefore the product is 2208184.

$$\begin{array}{r} 276023 \\ 8 \\ \hline 2208184 \\ \hline \end{array}$$

I shall now insert two or three examples like these for you to look over, and shall then *prove* the work of them to be correct by actual addition.

462508	7841902	37582431
4	6	9
1850032	47051412	338241879

462508	7841902	37582431
462508	7841902	37582431
462508	7841902	37582431
462508	7841902	37582431
	7841902	37582431
1850032	7841902	37582431
		37582431
	47051412	37582431
		37582431
		338241879

(21.) You thus see the great advantage of the above rule for multiplication, and how many figures and how much trouble are saved by it; you will observe that, throughout, the figures carried in the additions, from column to column, are the very same as the figures carried, from product to product, in the multiplications; and I think that nothing more need be said in the way of explaining and proving the rule.

(22.) I shall only remind you, that whenever you have to multiply a number by 10, all you will have to do is, to put a 0 to the number on the right; thus, 10 times 324 is 3240; 10 times 5237 is 52370, and so on. I say I have only to *remind* you of this, for you already know, from the Numeration Table, that by putting a 0 to a number, you push the units-figure of that number into the place of *tens;* the tens-figure into the place of *hundreds*, and so on: that is, every figure of the number is made ten times as great. And it is equally plain, that by putting *two* 0's, you multiply the number by 100; that by putting *three* 0's, you multiply by 1000, and so on.

Exercises.

1. Multiply 342 by 3.
2. Multiply 4761 by 4.
3. Multiply 7065 by 5.
4. Multiply 80724 by 6.
5. Mult. 1139509 by 7.
6. Mult. 273 by 12.
7. Mult. 75200564 by 8.
8. Mult. 9264073128 by 11.
9. Multiply 650098203470 by 12.
10. Before the opening of the Great Exhibition, 8615 Ladies' Tickets, at two guineas each, were sold; and

10892 Gentlemen's Tickets, at three guineas each: how many guineas were received for all these tickets?

11. The *average* number of daily visits to the Exhibition, that is, the number of visits, one day with another, for the last six days, was 89319: what was the total number of visits in the week?

12. The emigration from Ireland to the United States of America is at present (1851), on the *average*, more than a thousand persons a day: if the number of emigrants were exactly a thousand a day, how many persons would leave Ireland in a year or 365 days?

13. The greatest number of visitors to the Exhibition, on a five-shilling day, was 44512; this was on May 24th, 1851: the smallest number, on a five-shilling day, was 9327, on July 19th: how many shillings were received on both these days together?

14. Sound is borne to our ears by waves of air, produced by the sounding body. It moves at the rate of 1125 feet in a second; but light moves at the rate of about 192500 *miles* in a second; so that from the distance of a few miles, light may be said to reach the eye at the instant it appears: suppose then, 1st, that you observe a man breaking stones on a road, and that two seconds after seeing the fall of the hammer, you hear the blow: how many feet is he off? And 2nd, suppose that seven seconds after you see the flash of a cannon you hear the report: how many feet off is the gun?

15. Eleven seconds after a flash of lightning is seen the thunder is heard: how many feet off is the thunder-cloud?

You see, therefore, that the more quickly the thunder follows the lightning, the nearer the cloud is, and therefore the greater the danger. You may calculate the distance pretty nearly by counting the beats of your pulse, allowing 1000 feet of distance for the time between every two beats; for this time is in general a little less than a second. In a healthy person the pulse beats about 70 times in a minute.

16. In the week ending February 21, 1851, the number of Letters, sent through the Post-Office, was seven million two hundred and thirty-nine thousand nine hundred and sixty-two: at this rate, how many Letters are sent in twelve weeks?

(23.) The sign for multiplication is a cross of this form, $\times$,

placed between the two numbers to be multiplied together: thus, 4×6, means 4 *multiplied* by 6, or 6 multiplied by 4; you may say whichever you please, as the product is the same, namely 24; that is, $4 \times 6 = 24$. In like manner, $7 \times 9 = 63$, and $7 \times 9 \times 2 = 126$. The numbers which, multiplied together, produce another number, are called *factors* of that other number: thus, 3 and 4 are *factors* of 12; so are 6 and 2; and so are 2, 2, and 3. There are some numbers, such, for instance, as the number 13, that cannot be produced from *factors:* it is true that $13 \times 1 = 13$; but 1 is not considered to be a factor. Numbers of the kind now spoken of are called *prime numbers;* and all others, that is, all numbers that *can* be produced from factors, are called *composite numbers:* thus, 9, 12, 14, 16, &c., are *composite* numbers; but 7, 11, 13, 17, &c., are *prime* numbers. The products in the multiplication table are, of course, all composite numbers; indeed, you see that *every product* must be a composite number.

(24.) I need scarcely tell you, that what is here said of the two kinds of numbers, applies to *whole* numbers only; not to *halves*, *quarters*, &c.: three and a half, multiplied by 2, will produce 7; but 7 is not a composite number on this account, because three and a half is not a *whole* number; a *whole* number is also called an *integer;* the numbers used hitherto in this book are all *integers;* there are no *fractions*, as *halves*, *quarters*, &c., are called.

2. *When the Multiplier is greater than* 12.

RULE 1. Place the multiplier under the multiplicand, *units* under *units*, *tens* under *tens*, and so on.

2. Begin by multiplying by the *units-figure* of the multiplier, and you will get a row of figures, as in the former case; then, multiply, in like manner, by the *tens-figure*, taking care to put the first figure you get under that tens-figure, and you will thus have a second row of figures; then, multiply by the *third*, or *hundreds-figure*, of the multiplier, taking care, as before, to put the first figure you get in the new row, under that third figure: and proceed in this way till you have multiplied by every figure of the multiplier.

3. Add up all the rows of figures, and you will get the *product.*

For example: suppose you have to multiply 658 by 43.

Placing the multiplier 43 under the multiplicand, as in the margin, you would first multiply by the 3, and say, 3 times 8 are 24, 4 and carry 2; 3 times 5 are 15, and 2 are 17, 7 and carry 1; 3 times 6 are 18, and 1 are 19. The first row is now completed.

$$\begin{array}{r} 658 \\ 43 \\ \hline 1974 \\ 2632 \\ \hline 28294 \\ \hline \end{array}$$

You would then multiply by the 4, and say, 4 times 8 are 32, 2 and carry 3, and you would be careful to put the 2 under the figure you are multiplying by, that is, under the 4; 4 times 5 are 20, and 3 are 23, 3 and carry 2; 4 times 6 are 24, and 2 are 26. The second row is now completed; and as there are no more figures in the multiplier, you would draw a line under the two rows, and add them up: you would thus find the product of the two *factors*, 658 and 43, to be 28294. If, after having multiplied by the 3 and the 4, there had been still another figure in the multiplier, you would have had a third row of figures to add up: thus, if the multiplier had been 243, the work would have been as here shown: so that $658 \times 243 = 159894$. The next example in the margin is one in which the multiplier has *four* figures: you should look over the work; but you cannot want any explanation of it, after what has already been said. Below are three examples, similar to these in the margin, for you to work yourself.

$$\begin{array}{r} 658 \\ 243 \\ \hline 1974 \\ 2632 \\ 1316 \\ \hline 159894 \\ \hline \end{array}$$

$$\begin{array}{r} 378024 \\ 5346 \\ \hline 2268144 \\ 1512096 \\ 1134072 \\ 1890120 \\ \hline 2020916304 \\ \hline \end{array}$$

1. 764×35. 2. 764×356. 3. 242635×3456.

(25.) I have put these three examples here, rather than among the exercises at page 24, because I think it better that you should have a little practice in the Rule, before I show you how you may shorten the work in certain cases, I mean in those cases where noughts or ciphers occur in the multiplier or at the end of the multiplicand.

1. If ciphers occur at the end of the multiplier, and you were to proceed exactly as the rule tells you, you would get so many *rows* of ciphers: now you avoid the trouble of writing down these rows, by departing so far from the rule, as to push the ciphers as many places to the right hand. Thus, suppose you had to multiply 3264 by 2300: then, pushing the two

$$\begin{array}{r} 3264 \\ 2300 \\ \hline 979200 \\ 6528 \\ \hline 7507200 \\ \hline \end{array}$$

noughts, or *zeros*, two places to the right, you get the product by multiplying as in the margin; the two noughts being brought down before you begin.

2. If ciphers occur at the end of the multiplicand, you push them to the right in the same way: thus, in multiplying 372000 by 36, you work as in the margin, bringing down the three noughts before you begin. When *both* factors end in noughts, you bring down *all* the noughts before you begin to multiply, as in the second example in the margin, where the two factors, or numbers to be multiplied together, are 283000 and 470.

$$\begin{array}{r} 372000 \\ 36 \\ \hline 2232000 \\ 1116 \\ \hline 13392000 \\ \hline \end{array}$$

$$\begin{array}{r} 283000 \\ 470 \\ \hline 19810000 \\ 1132 \\ \hline 133010000 \\ \hline \end{array}$$

You plainly see, in each of these cases, that the noughts brought down before you begin the multiplication, may be left where they are till after you have done it, and may *then* be brought down to complete the product; but if you did not bring them down at first, it *might* happen that you might forget them: you can leave them till the last if you like.

3. If a cipher occur in the multiplier, anywhere except at the end, it is to be neglected altogether; for it could only give you a row of noughts, to be added to other figures, and the adding of noughts is useless. Thus, if we multiply 2473 by 3502, as in the margin, we put down a row of noughts, which has no effect upon the sum of the rows. The multiplier nought should therefore have been passed over. Remember, however, that whether you pass over noughts or not, the first figure, on the right, in each row, must be directly under the multiplying-figure that produces that row. If the row of noughts had been omitted in the last example, the 5, in the row 12365, must still have been placed under the multiplying-figure 5, in the multiplier 3502.

$$\begin{array}{r} 2473 \\ 3502 \\ \hline 4946 \\ 0000 \\ 12365 \\ 7419 \\ \hline 8660446 \\ \hline \end{array}$$

(26.) It only remains for me now to prove to you the truth of the rule for multiplication. For this purpose, let us refer to one of the preceding examples; the second example, for instance, at page 21, and let us examine what has actually been done in the working of it by the rule. The example is to multiply 658 by 243. Now 243 is 200, 40, and 3; so

that if we multiply 658, first by 200, then by 40, then by 3, and add up the three products, it is plain that we shall get the complete product required. The three *partial* products, that is, the products that are only *parts* of the whole product, are

$$\left.\begin{array}{r} 658\times 200 = 131600 \\ 658\times \;\;40 = \;\;26320 \\ 658\times \;\;\;3 = \;\;\;1974 \end{array}\right\}\text{the partial products.}$$

See (22), p. 18. 159894 the complete product.

By comparing this with the work at page 21, you see that when the noughts are rubbed out, the partial products are the same in both cases, the only difference being, that *there* the least of these products is written down first, and the greatest last, while here the greatest stands first, and the least last. Let us take the next example, where the multiplier is 5346; that is, 5000, 300, 40, and 6; and let us here multiply by the smallest number first, namely by 6, then by 40, then by 300, and lastly by the largest number, 5000.

$$\left.\begin{array}{lcr} 378024\times 6 & = & 2268144 \\ 378024\times 40 & = & 15120960 \\ 378024\times 300 & = & 113407200 \\ 378024\times 5000 & = & 1890120000 \end{array}\right\}\text{partial products.}$$

See (22), p. 18. 2020916304 complete product.

(27.) You thus see that the first row of figures found by the Rule, is the product given by multiplying by the *units* of the multiplier; the next row is the product given by multiplying by the *tens* of the multiplier, the nought on the right being omitted; the next row is the product given by multiplying by the *hundreds*, the *two* noughts being omitted, and so on. And you see that by leaving out these noughts at the end of the *partial* products, no error can be introduced into the *complete* product, which remains the same, whether the noughts be removed or not. It is plain that such is the principle of the rule, whatever be the multiplier.

(28.) There is a method of trying whether the product of two numbers is correctly brought out which ought to be shown to the learner, though the truth of it cannot be satisfactorily proved without the help of algebra: it is called *the method of casting out the 9's*, and depends on a property belonging to all numbers, namely, that if any number be divided by 9,

the *remainder* will be the same as that which arises from dividing the *sum* of all the digits or figures by 9; thus, the number 3264 divided by 9, leaves 6 for remainder; and the sum of the digits, namely 15, divided by 9, leaves also 6 for remainder. The method alluded to is thus expressed :—

Add up the digits of the multiplicand, rejecting every 9 that occurs among them; and in adding together the *other* figures, one after another, reject 9 every time this addition gives a number not less than 9. Do the same with the figures of the multiplier; and then multiply together the final results thus obtained; cast out the 9's from the product, and note what is left. Proceed in the same way with the digits of the *answer*, or product, and if the result here obtained *differ* from that before noted, the work is *wrong;* if it agree with it, the work is *most likely* correct.

This is the same as actually *dividing* multiplicand and multiplier by 9, preserving the *remainder* left from each, then dividing the product of these remainders by 9, and noting the remainder left; which last remainder cannot differ from that left by dividing the product by 9, if the work be right. And by using any other divisor in this way, the same conclusion follows; if the work be so tried with the two divisors, 9 and 11, which are the most convenient, it is very unlikely indeed that it can be wrong, if it stand both tests.* The reason why 9 is chosen is on account of the above-mentioned property of the sum of the digits of a number. Let us apply this property to the work of Ex. 3, page 22.

Multiplicand	2473,	rem. after rejecting 9s,	7	to be multiplied.
Multiplier	3502,	,, ,,	1	
			7	rem. after rejecting 9
Product	8660446,	,, ,,	7	

The work may therefore be considered as correct.

Exercises.

1. 463×247. 2. 789×674. 3. 2345×896.
4. 67082×7034. 5. 82060×5831. 6. 34728×65900.
7. 807900×64300. 8. 250978×64007.
9. 76830450×2001650. 10. 3456789×9876543.
11. $372 \times 583 \times 261$.
12. At midnight, on the 12th of August, 1830, a whirlwind, or hurricane, visited the West Indies, and passed on to America, at the rate of 18 miles an hour; it travelled onwards for about 168 hours: how many miles did it go in this time? †

* The higher any divisor be, the greater the faith to be placed in the test, because the less is the likelihood that a false product will give the *same* remainder as the true one, on account of the wider range of *possible* remainders. If the divisor be 2, the remainder from a false product is as likely to be the *correct* remainder as not.

† This onward course of the storm, combined with the velocity of its rotation, gave it a motion of about 100 miles an hour.—See *Rudimentary Navigation*, p. 65.

13. The total number of passengers carried by the railways of the United Kingdom in the year 1847 was 47484134; the *average* distance travelled by each was 16 miles; how many miles were travelled altogether?
14. Light passes from the sun to the earth in about 493 seconds; find from this the distance of the sun. (See Exercise 14, page 19.)
15. The total number of passengers carried by the railways of the United Kingdom in the first half of the year 1850, was 31766503, and in the second half, 41087919; the average distance travelled by each person was 17 miles; how many miles were travelled altogether in the year 1850? *
16. The greatest number of whales ever captured in the northern seas in one season was 2018, taken in the year 1823. Estimating the oil produced from each whale to have been worth 212 pounds, what was the value of all the oil produced?
17. The length of the building in which the Great Exhibition was held, is 1851 feet, corresponding to the date of the year; upwards of 5000 articles were dropped in it by visitors; of these about 1851 were afterwards recovered by application to the police; among the lost articles were 271 pocket-handkerchiefs. If the *number* of handkerchiefs be multiplied by the *number* which expresses the date of the year, the product will be nearly equal to the *number* of pounds received, as stated at page 4: find the difference between that number and the product.
18. The gross earnings of the London and North-Western Railway for the first half-year of 1851, for passengers and goods, were at the rate of 2273 pounds for every mile; the whole distance travelled up and down was 518 miles: what was the total amount of earnings for the half-year; and as the expenditure was 735257 pounds, what were the *net* earnings?

It is scarcely necessary that I should tell you, that when your multiplier is a *composite* number, you may if you please multiply by the *factors* which compose it, one after another, instead of

* It has been found that, for the last few years, the *average* length of each passenger's trip, year after year, is, within a trifle, uniformly 17 miles; and that the average payment for each trip is always, within a trifle, 2*s*.

multiplying by the number itself: thus, instead of multiplying by 72, you may multiply first by 12 and then by 6, because $12 \times 6 = 72$; and instead of multiplying by 126, you may multiply by 7, 6, and 3, because $7 \times 6 \times 3 = 126$; but, in general, little if anything is gained by this; and it is often troublesome to discover what the factors of a high number really are.

DIVISION.

(29.) The rule for Division teaches us the way of finding how many times the smaller of two numbers is contained in the greater. We might do this by repeated subtractions of the smaller number; that is, by subtracting it from the greater number, then subtracting it from the remainder, then again from the second remainder, and so on, till we could subtract no longer; the number of times we had subtracted would be the number of times which the greater number contains the smaller. But this would often be very tedious, and the trouble is saved by the rule I am about to give; for, just as Multiplication is a short way of finding the result of repeated *additions* of the same number, so Division is a short way of finding the result of repeated *subtractions* of the same number.* You remember, that in Multiplication you called the two numbers you were working with *multiplicand* and *multiplier;* in Division you call the two numbers *dividend* and *divisor;* and the number of times that the dividend contains the divisor is called the *quotient.*

(30.) 1. *When the Divisor is not greater than* 12.

Rule 1. Put the divisor to the left of the dividend, with a mark of separation between the two.

2. By help of the multiplication table, find how many times the divisor is contained in the first figure of the dividend, or in the number expressed by the first *two* figures, if the first figure alone be smaller than the divisor, and write the quotient underneath, taking care to observe what is *over;* for the divisor *may* be contained a certain number of times in the leading figures, and leave something over.

3. Now go to the next figure of the dividend, and fancy

* In strictness, Division is this, and something more; the ends accomplished by Division will be more fully seen when *Compound Division* is explained.

what was over to be *prefixed* to it; find how many times the divisor is contained in the number you thus get, and carry what is over to the *next* figure of the dividend, as before, and proceed in this way up to the last figure of the dividend.

An example or two will make this rule plain: thus, suppose I had to divide the number 3456 by 4. Then having placed the dividend 3456, and the divisor 4, as directed, I should see at once that 4 cannot be contained in the first figure 3; so I should say 4 in 34, 8 times and 2 over; then, fancying the 2 to be put before the 5, I should say 4 in 25, 6 times and 1 over; and putting this 1 before the 6, I should say 4 in 16, 4 times: therefore the *quotient* is 864; that is to say, 4 is contained exactly 864 times in 3456. Only consider how many subtractions we should have to perform here, if it were not for this rule of Division!

```
4)3456
 ―――――
   864
 ―――――
```

As a second example, suppose we had to divide 835465 by 6. We should proceed thus: 6 in 8 once and 2 over; 6 in 23, 3 times and 5 over; 6 in 55, 9 times and 1 over; 6 in 14, twice and 2 over; 6 in 26, 4 times and two over; 6 in 25, 4 times and 1 over. Therefore, 6 is contained in 835465, 139244 times, and there is 1 to spare. I must say a word or two about this *remainder*.

```
6)835465
 ―――――――
  139244...1
 ―――――――
```

When you divide a number by 6, the quotient that you get is the *sixth part* of that number; the quotient tells you not only how many sixes there are in the number, but at the same time what is the sixth part of that number: thus, if you find that there are eight sixes in a number, then you know that 8 is the sixth part of it; or that there are six eights in it. There are 8 sixes in 48; so there are 6 eights in 48: that is, 8 is the sixth part of 48. In like manner, if you divide by 4, the quotient tells you how many *fours* there are in the dividend, or what is the *fourth* part of it; and so of any other divisor. All this must be plain. When you say there are 3 *tens* in a number, you mean that 10 times 3 make that number; but 10 times 3 are the same as 3 times 10, namely, 30; so that if there are 3 *tens* in the number, there must be 10 *threes* in it, and so on.

The quotient in the example above, namely, 139244, is the *exact* sixth part only of 835464, so that of the 1 over no sixth part has been taken; consequently, the true sixth part of 835465 is 139244 and *one-sixth*, that is, a sixth part of 1.

Such a part of 1 is written thus, $\frac{1}{6}$, and is read *one-sixth;* so that the complete quotient in the above example is $139244\frac{1}{6}$. Suppose the number 835465 were so many shillings to be equally divided among six persons; then the sixth part, namely $139244\frac{1}{6}$, would of course be the number of shillings due to each; and as a sixth part of 1 shilling is twopence, the share of each person would be 139244 shillings and twopence. You see if you were to take no notice of the *remainder*, you would wrong each of these six persons out of twopence. Always therefore take account of the *remainder*, and write it, as here, with the divisor below it, and a little line.

$$\begin{array}{r} 7)46539 \\ \hline 6648\frac{3}{7} \\ \hline \end{array}$$

If it were required to divide 46539 by 7, then taking account of the *remainder*, which is 3, the *quotient*, or true seventh part, would be $6648\frac{3}{7}$. If these were shillings, as before, the share of each of the seven persons would be, 6648 shillings and a seventh part of 3 shillings. Three shillings is 36 pence, and a seventh part is therefore 5 pence and a penny over. There is no coin so small as the seventh part of a penny, so that if the persons are very particular, they must buy something with this penny, and divide *that* among themselves; but I dare say they would prefer to give it to you for working the example.

You will now, I think, be able to understand the work of the following examples, and to do the exercises on the next page yourself.

8)538641	9)725432	11)340261	12)1046285
$67330\frac{1}{8}$	$80603\frac{5}{9}$	$30932\frac{9}{11}$	$87190\frac{5}{12}$

NOTE 1. The sign for division is $\div$, meaning *divided by;* it is placed after the dividend, and before the divisor: thus, $12\div3$ means 12 *divided by* 3; so that $12\div3=4$; $24\div8=3$, &c. But division is otherwise represented by writing the divisor *below* the dividend with a little line between them, thus, $\frac{12}{3}=4$; $\frac{24}{8}=3$, &c.

NOTE 2. If you divide any number by 10, the quotient will be the very same as the number itself, *omitting the last figure*, which last figure will be the *remainder*, as you can easily see. Hence you may always write down the quo-

tient at once without any work: thus, $627 \div 10 = 62\frac{7}{10}$; $\frac{743}{10} = 74\frac{3}{10}$; $\frac{3529}{10} = 352\frac{9}{10}$; and so on: $\frac{7}{10}$, $\frac{3}{10}$, $\frac{9}{10}$, $\frac{2}{5}$, &c., are called *fractions*; they are read seven-tenths, three-tenths, nine-tenths, two-fifths, &c. The arithmetic of fractions will be fully explained hereafter.*

Exercises.

1. Divide 3724 by 5.
2. Divide 72081 by 6.
3. Divide 109234 by 7.
4. Divide 2006383 by 8.
5. Divide 52094100 by 9.
6. Divide 11380625 by 11.
7. Divide 10792039 by 12.
8. What is the value of $265837 \div 4$?
9. What is $\frac{872371}{7}$ equal to?
10. What is $\frac{34205}{5} + \frac{1863}{9}$ equal to?
11. What is $\frac{24603705}{8} - \frac{113223}{11}$ equal to?
12. What was the *average* number of persons daily visiting the Great Exhibition during the week commencing on Monday, Oct. 6, and ending on Saturday, Oct. 11? See Ex. 11, p. 8.†
13. Ireland contains four provinces: what was the average population of each province in the year 1851? See Ex. 8, p. 12.
14. Five years were spent on the work of the Britannia Tube over the Menai Straits; what was the average yearly expense? See Ex. 4, p. 4.
15. How many letters were on the average sent to the Post-Office every day of the week ending February the 21st, 1851? See Ex. 16, p. 19.
16. Letters are sometimes put into the Post-Office, by mistake, without any direction; others are so badly directed, that the postmen cannot find out where to take them; and in some cases the people to whom they are directed have removed. All these letters are sent

* I have thought it pardonable to say thus much about fractions *here*; this slight allusion to them is unavoidable, if the learner is expected thoroughly to understand what he is about.

† In these questions the *fractional* parts of the quotients are not to be considered.

to what is called the Dead Letter Office; many of these contain bank-notes and coin; many others contain money-orders, bankers' cheques, &c. The amount in bank-notes and cash found in the letters sent to the *Dead Letter Office, in the two years between January 5,* 1849, and January 5, 1851, was eighteen thousand eight hundred and seventy pounds; and the amount in money-orders, cheques, &c., was one million two hundred and twenty-six thousand two hundred and eighty-three pounds: * what was the average amount of the whole *quarterly* during these twenty-four months?

(31.) 2. *When the Divisor is greater than* 12.

Rule 1. Put the divisor on the left hand of the dividend, as in the former case, and mark off a place for the figures of the quotient on the right hand.

2. In order to get the first figure of the quotient, look at the first or leading figure of the divisor, and at the first or leading figure of the dividend: you will thus be able to see how often the former figure is contained in the latter, if this latter be the greater of the two; but if it be not the greater, then see how often the leading figure of the divisor is contained in the number formed by the first two figures of the dividend; the quotient you thus get is to be put down as the first figure of the complete quotient.

3. Multiply the divisor by this first figure, and subtract the product from the number formed by the corresponding leading figures of the dividend, and you will get the *first remainder:* but if it should happen that you cannot subtract, on account of the product being too great, it will be a proof that your quotient-figure is *too great;* you must therefore take it smaller, and begin again. If, on the other hand, you *can* subtract, and you find the remainder to be a number *not less* than the divisor, this will be a proof that your quotient-figure is *too small;* you must then begin again with a larger one. Your quotient-figure will often be too large if you neglect to take account of the *carryings* in multiplying the divisor by it, and it will sometimes be too small if you allow too much for these carryings.

* If the owners of this money do not apply for it within three years, the cash goes to the Revenue; the orders, bills, &c. are destroyed, and property of other kind is sold by auction for the benefit of the Revenue.

4. Having got the first remainder, which, mind, to be correct, must always be *less than the divisor*, annex to it the *next* figure of the dividend, and see how often the divisor is contained in the new number thus got; you will in this way find the second figure of the quotient; multiply the divisor by it, and, as before, get a second remainder, annexing to it the *next* following figure of the dividend; and proceed in this way till all the figures of the dividend have been used or brought down.

If any remainder should be so small, that, even after putting the proper dividend-figure against it, it be *less* than the divisor, the proper quotient-figure will, in that case, be 0, and *another* dividend-figure is to be brought down.

Ex. 1. Divide 315281 by 23.

```
23)315281(13707
   23
   ——
    85
    69
    ——
    162
    161
    ———
      181
      161
      ———
       20
      ———
```

Here, placing the dividend and divisor as directed, we see that the first figure of the divisor is contained *once* in the first figure of the dividend: we therefore put 1 in the quotient's place, and say, once 3 is 3, and once 2 is 2; and subtracting this 23 from the 31 above it, we get, for the first remainder, 8, to which we annex 5, the next figure of the dividend. We now have to see how often 23 is contained in 85, or, as learners usually say, how often 23 will *go* into 85. Looking, as before, only to the first figure of each number, we see that 2 will go into 8, 4 times: but a glance at the second figure of the divisor shows us that, in multiplying by this, we should have something to carry, so that the product of 23 by 4 would be greater than 85. This warns us that 4 is too great: we therefore put 3 for the second quotient-figure; and, multiplying the divisor by this 3, and subtracting the product, we get 16 for the second remainder; and bringing down the next figure, 2, of the dividend, we have to see how often the divisor goes into 162. As the first figure, 1, of this number is less than the first figure, 2, of the divisor, we now have to see how often the 2 goes into 16; this we at once find to be 8 times: but we at the same time know that 8 must be too great, on account of what we should have to carry after multiplying the 3 by it; so we put 7 for the third quotient-figure, multiply the divisor by it, subtract the product from

162, and thus get a third remainder, 1, which becomes 18 when the next dividend-figure is annexed to it. We now have to see how often 23 goes into 18; and as it plainly goes *no times*, we put a 0 for the fourth figure of the quotient, and bring down the next figure of the dividend; so that we have to find, lastly, how often 23 goes into 181: the 2 will go into the 18, 9 times; this, however, we know to be too great, because of what must be carried from 9 times 3: we therefore try 8. The product of 23 by 8 is 184, which being greater than 181, we reject the quotient-figure 8, as too large, rub out the product by it, and try 7. The product of 23 by 7 is 161, which is less than 181; 7 is therefore the correct fifth quotient-figure, and the final remainder is 20, with which remainder the work ends, as all the figures of the dividend have been brought down.

The final remainder is to be treated just as the last remainder in the former rule: it is to be written against the quotient, with the divisor underneath, and a short line between them; the complete quotient is therefore $13707\frac{20}{23}$, the *fraction* at the end being *the twenty-third part of* 20, so that the twenty-third part of the number 315281 is 13707, and the twenty-third part of 20 besides.

After the full explanation here given of every step of the work of this example, I think you will be able to understand the operations below without any further help.

75)826052($11014\frac{2}{75}$	237)14658293($61849\frac{80}{237}$
75	1422
76	438
75	237
105	2012
75	1896
302	1169
300	948
2	2213
	2133
	80

```
4652)327006413(70293 3377/4652
     32564
     -----
      13664
       9304
      -----
       43601
       41868

        17333
        13956
        -----
         3377
```

Whenever your *divisor* ends in *ciphers* or *zeros*, cut them off, or fancy them rubbed out, and cut off *as many* figures from the *right hand* of the dividend: then proceed with the work just as you would do if neither the ciphers nor the figures cut off were there; but remember, that when you arrive at the *last* remainder, you must put against it *all* the figures you have cut off from the dividend to get the *complete* remainder. Suppose, for instance, you had to divide 278643 by 3500, and that you allowed the *noughts* to remain, the work would stand as in the margin, as you already know, where you see that the last *two* figures of the dividend, namely 43, are also the last two figures of the remainder. Now if you had cut these two figures off, as also the two ciphers, and had taken no notice of them till the end, you would have got, by uniting the two dividend-figures to the last remainder, the same quotient and the same remainder that you have got now, as the margin shows: you see, therefore, that by following this plan, you do away with useless ciphers. The complete quotient in this example is $79\frac{2143}{3500}$.

```
3500)278643(79
      24500
      -----
       33643
       31500
       -----
        2143
       -----

35\00)2786\43(79
       245
       ---
        336
        315

         2143
         ----
```

```
237)14658293(60000
    14220000
    --------
237)  438293(1000
      237000
      ------
   237)201293(800
       189600
       ------
   237) 11693(40
         9480
         ----
     237)2213(9
         2133
         ----
           80
          ---
          237
```

(32.) I have now to explain to you the principle of the rules for division, and to show you that they always lead us to the true quotient. Let us return to the second example above, and see what has been done: this example is to divide 14658293 by 237; that is, we are required to find the 237th part of 14658293. By comparing the work with that in the margin, you will see that we have done what was required, by finding first the 237th part of a portion of the number, then the 237th part of another portion of it, then of another, and so on, till all the portions of it have been divided. You see that the several portions of the number here are 14220000 + 237000 + 189600 + 9480 + 2133 + 80 = 14658293. The 237th part of the *first* portion is 60000; the 237th part of the *second* portion is 1000; the 237th part of the *third* portion is 800; of the *fourth* portion, 40; of the *fifth*, 9; and the 237th part of the *last* portion is $\frac{80}{237}$; so that the 237th part of the *whole* is $61849\frac{80}{237}$. And it is plain that a similar explanation applies to every example worked by the rule.

In order to *prove* whether any operation in division is correct, you have only to multiply the quotient (without the fraction) and the divisor together, and to add the last *remainder* to the product; the result will be the dividend, if the work be right. Thus, if in the example just examined, 61849 be truly the 237th *part* of 14658293, with 80 over, then 237 *times* 61849, with 80 taken in, must of course be 14658293.

(33.) The rule I have just been explaining is called the rule of *Long Division;* the former rule, which serves only for divisors not greater than 12, is called the rule of *Short Division.* Examples in *this* rule may of course be also worked by long division; in fact, the second rule includes the first, only that examples in the first may be worked in less space. Nobody would actually do such examples by *long* division; and I only mention, that the second rule really includes the first, lest you should be in any doubt as to

whether the principle of short division ought to be separately explained. Rather than work a short-division example by long division, it will be wiser to work a long-division example by short division, as may always be done when the divisor can be expressed by *factors*, of which no one is greater than 12: thus, 56 is composed of two such factors, for $8 \times 7 = 56$. If, therefore, you have to divide a number by 56, you may first divide the number by one of these factors, and then divide the quotient by the other: both forms of division are shown in the margin.

```
        56)38248(683
           336
           ---
8)38248     464
-------     448
7) 4781
-------      168
    683      168
    ---      ---
```

This is an example with no remainder: in some instances, the division by the first factor leads to a remainder, the division by the second factor to another remainder, and so on. Now when division by any factor leaves a remainder, you must write a *fraction*, as in the former examples, having this remainder above, and the number you have divided by below. Should the next factor also leave a remainder, you must multiply *this* remainder by the lower number of the former fraction, and add in the upper number of it; this will give the *upper* number of the next fraction, and the product of the divisors already employed will be the *lower* number of the new fraction. Two examples are given in the margin: in the first of them, the divisors are 8 and 7, the factors of 56; in the second, the divisors are 3, 6, and 7, the factors of 126; that is, $3 \times 6 \times 7 = 126$. If you use long division for the two *composite* divisors, 56 and 126, you will find the quotients to be the same. I cannot explain to you *now* how it happens that the *fractional* part of each quotient is truly brought out, as in the margin, but you will see it all very clearly when you come to *Fractions*.

$8)38259$

$7)\ 4782\frac{3}{8}$

$683\frac{11}{56}$

$3)24632$

$6)\ 8210\frac{2}{3}$

$7)1368\frac{8}{18}$

$195\frac{62}{126}$

Exercises.

1. Divide 2463 by 47.
2. Divide 39072 by 83.
3. Divide 1197054 by 342.
4. Divide 8264921 by 576.

5. Divide 9460257 by 843. 6. Divide 40627385 by 2712.
7. Divide 7926400 by 3046.
8. Divide 81729361 by 402600.
9. What is $17690234 \div 78910$?
10. What is $27506381 \div 80247$?
11. What is the value of $\frac{59264743}{26083}$?
12. What is the value of $\frac{1079256401}{3427500}$?
13. The gross earnings of the Great Western Railway, for passengers and goods, for the first half of the year 1851, were 438834 pounds; the distance travelled up and down is 264 miles: what were the earnings per mile for the half-year?*
14. It is found that in a large number of persons, say a hundred thousand, aged 25 years, about one in every 26 dies before arriving at the age of 30 years: how many may be expected to die out of the whole in these five years?
15. It is estimated that 1 out of every 1585 persons in Great Britain is deaf and dumb: how many deaf and dumb persons are there in the entire population, which, according to the *Census* of 1851, is 20936468 persons?
16. The number of blind persons in Great Britain is at the rate of about one in every thousand: how many more blind persons are there than deaf and dumb?
17. The Himalaya mountains in India are the highest in the world; some of them have been found to measure twenty-seven thousand feet: how many buildings, as high as St. Paul's in London, must be piled, one upon another, to reach to this height, the height of St. Paul's being 344 feet?
18. The shilling catalogue of the Great Exhibition contains 320 pages, and is more than half an inch in thickness: there were three hundred thousand copies printed: suppose that each copy were pressed to the thickness of half an inch, and that all were then piled up, one upon another, how many times the height of the Monument, near London Bridge, would the pile reach, the height of the Monument being 202 feet, or 2424 inches?

* In these questions the *fractional* parts of the quotients need not be taken into account.

(34.) You have now been brought through the first *four* rules of Arithmetic: from the explanations which have been given of them, I think you should understand the *reasons* of the operations those rules direct to be performed: and from the many exercises which have been proposed, you ought to be pretty ready in the *practice* of them. Everything that follows will depend upon these four rules: there are no operations in arithmetic besides those of addition, subtraction, multiplication, and division. Different names will be given to different rules in the remaining part of the subject; but adding, subtracting, multiplying, or dividing, are the only operations that can enter into the work of any of them. The rules now given are called the four *simple* rules of arithmetic; *simple* addition, *simple* subtraction, *simple* multiplication, and *simple* division. They are called *simple*, not because they are so *easy*, but because every example belonging to them has to do simply with *one denomination* of quantities or things, as *pounds*, *persons*, *miles*, &c. These are each called things of *the same denomination*. A sum of money, composed of so many pounds, so many shillings, and so many pence, is composed of quantities of *different denominations;* and in like manner a distance or length composed of miles, yards, and feet, is composed of quantities of different denominations; and accordingly, when the rules of addition, subtraction, &c., are applied to *these*, they are called *compound* rules, so that you see the cause of the distinction between *simple* addition and *compound* addition, between *simple* subtraction and *compound* subtraction, &c.

(35.) I am now going to introduce you to the arithmetic of compound quantities; but before I do so, you will have to learn a few *tables*, in order that you may know how many quantities of a lower denomination are contained in a single quantity of the same kind, but of a higher denomination; as, for instance, how many feet in a mile, how many ounces in a pound weight, how many gallons in a barrel; and so on.

The table of liquid measure, at page 42, contains more particulars than you need commit to memory, as the alteration in the capacity of the gallon, which took place in 1826, alters the *number* of gallons now contained in the old casks. The *imperial gallon* is used equally for wine and malt liquor; it is less than the old beer gallon, and greater than the old wine gallon. (See foot-note, page 42.)

TABLES OF MONEY, TIME, WEIGHTS, MEASURES, &c.

MONEY.

In the United Kingdom, money-accounts are kept in *pounds*, *shillings*, *pence*, and *farthings*. The symbols or marks used for these are £, for pounds; *s*, for shillings; and *d*, for pence: *q* is sometimes used for farthings, or *quarters* of a penny, and sometimes the letter *f*. The pound used to be represented by a Bank-note, value 20*s.*; it is now replaced by a gold coin, called a *sovereign;* there was formerly another gold coin, called a *guinea*, the subdivisions of which were the *half-guinea*, and the *seven-shilling piece.* The value of the guinea was 21*s.* Though this latter coin has gone out of circulation, it is still customary to call 21*s.* a guinea. The weights of this coin and its subdivisions are given below; as also of the 5*s.* piece, or *crown*, and *half-crown*, in the present coinage.

Gold Coins—*Old Coinage.*

Name.	*Value.*	*Weight, troy.*	
Guinea	21*s.* 0*d.*	5 dwt.*	9 $\frac{39}{89}$ gr.
Half-Guinea	10*s.* 6*d.*	2	16 $\frac{64}{89}$
7*s.* Piece	7*s.* 0*d.*	1	19 $\frac{13}{89}$
	New Coinage.		
Sovereign	20*s.*	5	3 $\frac{171}{623}$
Half-Sovereign	10*s.*	2	13 $\frac{397}{623}$
	Silver Coins—*New Coinage.*		
Crown	5*s.* 0*d.*	18	4 $\frac{4}{11}$
Half-Crown	2*s.* 6*d.*	9	2 $\frac{2}{11}$
Shilling	12*d.*	3	15 $\frac{3}{11}$
Sixpence	0*s.* 6*d.*	1	19 $\frac{7}{11}$
4*d.* Piece	0*s.* 4*d.*	1	5 $\frac{1}{11}$
3*d.* Piece	0*s.* 3*d.*	0	21 $\frac{9}{11}$

* The mark *dwt.* means pennyweight, and *gr.* stands for grains.—See table of Troy Weight.

Pence Table.

	s.	d.	Pence.		s.	d.	Pence.		s.	d.
4 farthings make		1	50	make	4	2	100	make	8	4
12 pence	1	0	60	„	5	0	108	„	9	0
20 „ „	1	8	70	„	5	10	110	„	9	2
24 „ „	2	0	72	„	6	0	120	„	10	0
30 „ „	2	6	80	„	6	8	130	„	10	10
36 „ „	3	0	84	„	7	0	132	„	11	0
40 „ „	3	4	90	„	7	6	140	„	11	8
48 „ „	4	0	96	„	8	0	144	„	12	0

A farthing is represented thus: $\frac{1}{4}d.$, meaning one-fourth of a penny; two farthings, or one halfpenny, thus: $\frac{1}{2}d.$; and three farthings thus: $\frac{3}{4}d.$

NOTE.—Gold coins are allowed by law to pass under the full weight, an allowance of a little more than $\frac{1}{2}$ grain being made for the diminution of weight by use: thus, a sovereign passes for its full value, provided it reach the weight of 5 dwt. $2\frac{3}{4}$ gr. Gold coins are not *wholly* of pure gold; they are made of what is called *standard* gold, which is composed of 11 parts of pure gold, and 1 part of an *alloy* of either pure copper or copper and silver. Any quantity of gold, whether alloyed or not, is, for convenience, supposed to be divided into 24 parts, called *carats;* and the degree of purity of the mass is expressed in these carats: thus, standard or sterling gold is 22 *carats fine*, the remaining 2 carats being alloy. Unalloyed gold is 24 carats fine. Gold is not alloyed in coinage from motives of frugality, but for convenience of workmanship, and for the purpose of rendering the coin harder and more durable. Articles of plate, in gold, are allowed to be of standard purity,—that is, 22 carats fine; but in watch-cases this degree of fineness is not permitted; the standard for *them* is 18 carats fine; so that one-fourth of the material is alloy. The "Hall mark," which guarantees the proper degree of purity, is usually stamped on the ring that suspends the watch, and on other parts of the case.

Standard silver, for silver coins, contains 11 oz. 2 dwts. of pure silver in 1 lb. troy, and 18 dwts. of alloy.

TIME.

60 seconds,	marked thus,	60 sec.	make	1 minute.
60 minutes,	„	60 m.	„	1 hour.
24 hours,	„	24 h.	„	1 day.
7 days,	„	7 d.	„	1 week.
52 weeks 1 day	„	52 w. 1 d.	„	1 year.
52 weeks 2 days, or 366 d.			„	1 leap year.

Leap year occurs once in every *four* years, when a day must be added to 365, the number of days in a *common* year; the month which takes this additional day is *February*, which in leap year has 29 days. If the

date of the year be divided by 4, and there be no remainder, that year will be leap year; and 1, 2, or 3, will remain, according as the year is 1, 2, or 3 years after leap year. The number expressed by the last two figures only of the date may be used instead of the complete date, as the remainder arising from division by 4 will be the same: thus, the year 1852 is leap year, since 52 is exactly divisible by 4. The addition of a day every fourth year is rendered necessary on account of the ordinary year being taken as 365 days, instead of 365¼ days, which it is within a few minutes; so that in 4 years a whole day would otherwise be overlooked. Even as it is, the few minutes just noticed, by which 365¼ days differs from the true year, as shown by the sun, accumulate to an error of about 3 days 2½ hours in 400 years. This error is an error of *excess;* for the true solar year is 365 d. 5 h. 48 m. 49 sec. To remove the effect of this error, it has been fixed that when the year consists of complete *centuries* (a century being 100 years), although the date would be exactly divisible by 4, yet the year is not to be considered as leap year, unless the date, omitting the *two* final *noughts* or zeros, is also divisible by 4: thus, the years 1800, 1900, 2100, &c., are to be reckoned as common years, since 18, 19, 21, &c., are not divisible by 4; but 1600, 2000, &c., are leap years. With this correction the civil reckoning so far agrees, on the average, with the astronomical determination of the year, as to be only one day in advance of the strict truth in 3546 years,—an error which for the purposes of life it is unnecessary to make any provision for.

The year is divided into twelve parts,—January, February, March, April, May, June, July, August, September, October, November, and December. These are called the 12 *calendar months.* Each of these, except February in a common year, contains more than 4 weeks, or 28 days; yet, in ordinary language, 4 weeks is called a month. There are $13\frac{1}{28}$ of *these* months in a calendar year. The number of days in each calendar month may be easily recollected by aid of the following lines:—

Thirty days have September,
April, June, and November,
February has twenty-eight alone,
And all the rest have thirty-one;
But leap-year, coming once in four,
February then has one day more.

WEIGHTS.

Avoirdupois.

			marks.
16 drams	make	1 ounce	(oz.)
16 ounces	„	1 pound	(lb.)
28 pounds	„	1 quarter	(qr.)
4 quarters, or 112 lbs.	„	1 hundred-weight	(cwt.)
20 hundred-weight	„	1 ton	(t.)

By this weight coarse and bulky goods are weighed, and all the common necessaries of life. The weight called a *stone* is also much used for like purposes; but it is not fixed, like the other weights given above; in general, however, by a stone is meant 14 lbs. avoirdupois. In London, a stone of butcher's meat is only 8 lbs.; but in many country places it is 14 lbs., and in some 16 lbs. Unless, however, the contrary be stated (butchers' meat excepted), by a *stone* 14 lbs. is always understood, so that 8 stone make a cwt.; 2 stone of 14 lbs. make 1 tod of wool; 6½ tods, 1 wey; 2 weys, 1 sack; and 12 sacks 1 last: 12 sacks also make a chaldron of coals,—a measure now discontinued.

TROY.

			marks.
24 grains	make	1 pennyweight	(dwt.)
20 pennyweights	„	1 ounce	(oz.)
12 ounces	„	1 pound	(lb.)

This weight is for the precious metals, and for ingredients used in philosophical experiments. The grain troy is subdivided into 20 parts, called *mites*; so that 20 *mites* = 1 grain. The avoirdupois ounce is less than the troy ounce; for 1 oz. avoir. = $\frac{175}{192}$ oz. troy; but the avoirdupois pound is greater than the troy pound; for 1 lb. avoir. = $\frac{175}{144}$ lb. troy.

APOTHECARIES.

			marks.
20 grains	make	1 scruple	(sc. or ℈)
3 scruples	„	1 dram	(dr. or ʒ)
8 drams	„	1 ounce	(oz. or ℥)
12 ounces	„	1 pound	(lb.)

Apothecaries and chemists use this weight in mixing medicines; but drugs are bought and sold by avoirdupois weight.

MEASURES

OF LENGTH, SURFACE, AND SOLID OR CUBIC CONTENTS.

LENGTH (or LONG MEASURE).

			marks.
12 inches	make	1 foot	(ft.)
3 feet	„	1 yard	(yd.)
6 feet	„	1 fathom	(fath.)
5½ yards	„	1 rod, pole, or perch	(per.)
4 perches, or 100 links		1 chain (22 yards)	
40 poles	„	1 furlong	(fur.)
8 furlongs	„	1 mile (1760 yds.)	(mi.)
3 miles	„	1 league	(lea.)

Cloth.

2¼ inches	„	1 nail	(na.)
4 nails	„	1 quarter of a yard	
5 quarters	„	1 ell English	
3 quarters	„	1 ell Flemish.	

A *hand*, in measuring horses, is 4 in.; a *span*, 9 in.; and a *pace*, 5 feet. A rood of fencing or ditching is 7 yards.

Surface (or Square Measure).

144 square inches	make	1 square foot.
9 square feet	„	1 square yard.
30¼ square yards	„	1 sq. rod, pole, or perch.
40 sq. perches	„	1 rood.
4 roods	„	1 acre.
10 sq. chains, or 100000 sq. links		1 acre.
640 acres	„	1 square mile.
100 square feet	„	1 square of flooring.
272¼ square feet	„	1 sq. rod of brickwork.

Solid, or Cubic Measure.

1728 cubic inches make 1 cubic foot; and 27 cubic feet make 1 cubic yard.

Measures for Wine, Spirits, Ale, Beer, &c.

	4 gills	make	1 pint.
	2 pints	„	1 quart.
	4 quarts	„	1 gallon.
Wine.*	10 *gallons*	„	1 anker.
	18 *gallons*	„	1 runlet.
	31½ *gallons*	„	1 barrel.
	42 *gallons*	„	1 tierce.
	63 *gallons*	„	1 hogshead (hhd.)
	84 *gallons*	„	1 puncheon.
	126 *gallons* (2 hhds.)	„	1 pipe, or butt.
	252 *gallons*	„	1 tun, or 2 pipes.
Ale and Beer.*	9 *gallons*	„	1 firkin.
	2 firkins	„	1 kilderkin.
	2 kilderkins (36 *gal.*)	„	1 barrel.
	54 *gal.* or 1½ bar.	„	1 hogshead.
	72 *gal.* or 1⅓ hhd.	„	1 puncheon.
	108 *gal.* or 1½ pun.	„	1 butt.
	216 *gal.* or 2 butts	„	1 tun.

* These measures are inserted here chiefly because the learner may know the number of gallons meant whenever he meets with the names of them; but he must be here apprised, that these gallons are not *imperial* gallons, but gallons according to the *old measure*, which has been abolished. Old gallons, *wine* measure, are converted into imperial measure by multiplying by $\frac{5}{6}$; and old gallons, *ale* measure, are converted into imperial measure by multiplying by $\frac{59}{60}$, which fractions are sufficiently accurate for practical purposes. The names of the measures for wine and ale, marked above, are *now* merely the names of *casks*, and do not denote imperial *measures;* indeed, some of these names were never un-

The imperial gallon is a measure of the same uniform capacity, whether for wine, ale, beer, corn, or any other commodity; it must contain exactly 10 lb. avoirdupois of distilled water. The barrel, hogshead, &c. differs in capacity, according as it is used for wine or beer.

Corn, or Dry Measure.

2 pints	make	1 quart.	2 bushels	make	1 strike.
2 quarts	„	1 pottle.	2 strikes	„	1 coom.
2 pottles	„	1 gallon.	2 cooms (8 bush.)		1 quarter.
2 gallons	„	1 peck.	5 quarters	„	1 wey or load.
4 pecks	„	1 bushel.	2 weys	„	1 last.

The standard weight of a sack of coals is 2 cwt.; so that 10 sacks weigh 1 ton. A ship-load is 4240 sacks, or 8480 cwt. A sack contains 3 *bushels*, heaped measure; but heaped measure is now abolished.

Division of the Circumference of a Circle.

The circumference of every circle is supposed to be divided into 360 equal parts, called *degrees;* these, of course, are longer or shorter, according as the circle is greater or less. Each degree is divided into 60 equal parts, called *minutes;* and each minute into 60 equal parts, called *seconds:* the marks for degrees, minutes, and seconds, are a small $^\circ$ for degrees, a dash $'$ for minutes, and two dashes $''$ for seconds, thus:

$$60'' = 1'; \; 60' = 1^\circ; \; 360^\circ = \text{a whole circumference}; \; 90^\circ = \text{a quadrant}.$$

A degree of the circle round the earth at the equator, or a degree of a meridian, is about $69\frac{1}{15}$ miles; so that the length of $1'$ is the 60th part of this, which is the length of a *sea-mile*, or, as it is frequently called, of a *geographical mile;* a geographical, or nautical mile, being the length of $1'$ of the equator or meridian; it exceeds a land mile by about $\frac{1}{7}$ of that mile.

---

alterably fixed in meaning: a pipe of wine of one kind often differed considerably in measure from a pipe of another kind. Whatever *name* be still retained for the *cask*, the liquor contained in it is always *gauged* or *measured*, and valued in imperial gallons accordingly. It is useful to remember, however, that if a person were now to purchase a runlet of wine,—that is, a cask of that name full of wine,—he would get only $\frac{5}{6}$ of 18 imperial gallons; that is, only 15 gallons imperial measure. The word *gallons* is printed in italics above to imply that *old measure* is meant.

REDUCTION.

(36.) Reduction is the name given to the operations by which a quantity is reduced to another of the same value, but of different denomination. The operation, for instance, by which pounds in money are reduced to shillings, pence, or farthings; or farthings to pounds, years to hours, &c. You see, therefore, that Reduction is of two kinds: the reduction of a higher denomination to a lower, and the reduction of a lower to a higher; it is therefore comprised in two rules.

1. *To Reduce a Quantity to one of a Lower Denomination.*

Rule. See by the tables how many quantities of the *next* lower denomination make *one* of the higher, and multiply the proposed quantity by that *number;* the product will be the quantity in the next lower denomination.

If it is to be reduced still lower, see how many quantities of that next lower denomination make *one* of the denomination already reached, and, as before, multiply by that number; and so on till you reach the denomination required.

Ex. 1. Let it be required to reduce £124 to farthings.

As 20 *shillings* make *one* pound, we first multiply by the number 20; this reduces the £124 to 2480 *shillings;* and, since 12 *pence* make *one* shilling, we then multiply the number 2480 by the number 12, which reduces the 2480 shillings to 29760 *pence;* and, lastly, since 4 farthings make *one* penny, we multiply the number 29760 by 4, which finally reduces the £124 to 119040 farthings.

£124	
20	
2480	shillings.
12	
29760	pence.
4	
119040	farthings.

If any quantities of the lower denominations are connected with the quantity to be reduced, we must, of course, add them in with the products which give the *same* denominations; thus, if *shillings* had been connected with the £124 above, these shillings must have been added in with the product 2480, which gives *shillings;* and if *pence* had also been connected with the pounds, we must have added them in with 29760, the product which gives *pence;* and so on.

2. Suppose we had to reduce £124. 13*s*. 4½*d*. to farthings.

Then, multiplying the 124 by 20, and taking in the 13, we have 2493 shillings; and multiplying 2493 by 12, and taking in the 4, we have 29920 pence; and, lastly, multiplying 29920 by 4, and taking in the 2 farthings, we have finally 119682 farthings.

£.	*s*.	*d*.
124	13	4½
20		
2493 shillings		
12		
29920 pence		
4		
119682 farthings		

It is proper that I should notice here, that in reducing pounds to shillings you do not multiply the *pounds* by 20, but only the *number* of pounds; if *pounds* be multiplied by any number the product must be *pounds*. In like manner, in reducing to pence, it is not the *shillings* you multiply, but only the *number* of them. It would be tedious to be always making this distinction in rules and examples, though it is right that you should not be misled by the brief language in which rules are sometimes expressed.

3. Reduce £372. 15*s*. 7¾*d*. to farthings.

Here we multiply by 20, and take in the 15; then by 12, and take in the 7; and, lastly, by 4, and take in the 3; as in the margin.

£.	*s*.	*d*.
372	15	7¾
20		
7455 shillings		
12		
89467 pence		
4		
357871 farthings		

4. How many minutes are there in 29 days, 3 hours, and 21 minutes?

Since 24 hours make one day, and 60 minutes one hour, we have to multiply, first by 24, taking in the 3 hours, and then by 60, taking in the 21 minutes, as in the margin.

d.	*h*.	*m*.
29	3	21
24		
119		
58		
699 hours		
60		
41961 minutes.		

You will of course understand, when it is said that 24 hours make a day, that what in common language is called a day and a night is meant. People in general consider a day to end at 12 o'clock at night, and then a new day to commence, which lasts till 12 o'clock the following night, thus completing 24 hours; yet it is customary to call that part of the 24 hours usually devoted to sleep, *night*; and to apply the term *day* more especially to the other portion. Astronomers begin their day at noon, and end it at the noon following. What they would

call Jan. 5, at 18 h. 15 m., we should call Jan. 6, at a $\frac{1}{4}$ past 6 in the morning.

5. How many grains are there in a lump of gold, weighing 17 lb. 6 oz. 14 dwt. 21 gr.?

Here we have to multiply, first by 12, taking in the 6; then by 20, taking in the 14; and, lastly, by 24, taking in the 21. As the first figure, arising from multiplying by the 4, expresses *units*, we shall add the *units* in 21, namely the 1, to *this* figure; and as the figure arising from multiplying by the 2 is *tens*, we shall add the *tens* in 21, namely the 2, to *this* figure.

lb.	oz.	dwt.	gr.
17	6	14	21
12			
210			
20			
4214			
24			
16857			
8430			
101157 grains.			

NOTE. You may sometimes have to multiply by a number with a *fraction* joined to it, as, for instance, by $5\frac{1}{2}$, in order to reduce perches to yards; to do this you may first multiply by the 5, and then to the product add *half* the multiplicand; or you may multiply by *twice* $5\frac{1}{2}$, that is by 11, and then divide the product by 2. If the fraction be *one-fourth*, or *three-fourths*, you may reduce all to *fourths*, by multiplying the number to which the fraction is joined by 4, and taking the odd *fourths* in; the result will then be *four times* the true multiplier, which you may use instead of the true one; but then you must remember to divide the product by 4, to get the true product. In general, however, the best way will be, when you have to multiply by $\frac{1}{4}$, to take a fourth part of the multiplicand; and when you have to multiply by $\frac{3}{4}$, to take half the multiplicand for *two* fourths, and then half of this for the *remaining* fourth. These portions of the multiplicand, added to the product you get by using the multiplier *without* the fraction, will give the *complete* product. Thus: suppose 327 is to be multiplied by $5\frac{1}{2}$, and $30\frac{1}{4}$, respectively, the work is as follows:

2)327	4)327
$5\frac{1}{2}$	$30\frac{1}{4}$
1635	9810
$163\frac{1}{2}$ *for* $\frac{1}{2}$.	$81\frac{3}{4}$ *for* $\frac{1}{4}$.
Product $1798\frac{1}{2}$	Product $9891\frac{3}{4}$

In the first of these operations, 327 perches are reduced to yards; in the second, 327 square perches, or rods, are reduced to square yards.

Exercises.

1. Reduce £865. 17*s*. 5*d*. to pence.
2. Reduce £397. 16*s*. $4\frac{3}{4}$*d*. to farthings.
3. How many minutes are there in 365 days?
4. How many pounds are there in 5 cwt. 3 qr. 18 lb. of cheese?
5. Reduce 73 oz. 17 dwt. 11 gr. of gold to grains.
6. How many inches are there in $237\frac{1}{2}$ yards of length?
7. Reduce 47 miles 5 furlongs 9 perches 3 yards to yards.
8. How many square yards are there in 7 acres?
9. How many pounds are there in 3 tons 13 cwt. 2 qrs. 22 lb.?
10. Reduce 46 barrels of beer to quarts (*old measure*).
11. The middle arch of the Southwark Iron Bridge weighs about 1523 tons: what is its weight in pounds?
12. The great bell of St. Paul's weighs 5 tons 2 cwt. 1 qr. 22 lb.: what is its weight in pounds?
13. The largest bell in the world is that of Moscow; its weight is 192 tons 17 cwt. 16 lb.: reduce this to pounds.
14. The money taken in *silver* alone at the doors of the Great Exhibition weighed about 35 tons: how many avoirdupois ounces did it weigh; and how much silver money was taken, allowing 5*s*. to weigh an avoirdupois ounce, as is very nearly the case?
15. A pipe of wine is to be drawn off in an *equal number* of quart, pint, and half-pint bottles: how many of each will there be (*old measure*)?
16. How many grains are there in three dozen of table-spoons, each spoon weighing 2 oz. 4 dwt.?
17. The ground occupied by St. Paul's Cathedral measures 2 acres 16 perches: how many square feet are there in this extent?
18. A vat, or large cask for preserving beer, was built for Mr. Meux, the brewer, so capacious that 400 men stood without inconvenience inside of it: it held twelve thousand barrels of beer; how many quarts did it contain (*old measure*)?
19. The total receipts of the Great Exhibition were £505107: if this sum were reduced to shillings, and a person were to begin counting them as soon as Jan. 1, 1852, com-

menced, at the rate of 80 shillings a minute, and to continue counting twelve hours a day, in what month, and on what day of that month would he have finished his wearisome task?* (Remember that 1852 is leap-year.)

20. The weight of the gold taken at the doors of the Great Exhibition was about one thousand seven hundred and fifty-one pounds troy: how many sovereigns were there, each sovereign weighing 123 grains?

(37.) 2. *To Reduce a Quantity to one of Higher Denomination.*

RULE 1. Reduce the quantity to the *next* higher denomination, by dividing it by the *number* which expresses how many of the lower denomination make *one* of the higher.

2. In like manner, reduce this new denomination to that next higher, by again dividing; and so on till the proposed denomination is reached.

Ex. 1. Reduce 119040 farthings to pounds.

First, dividing by 4, to reduce the farthings to pence, we get 29760 pence; next, dividing by 12, to bring these pence into shillings, we reduce it to 2480 shillings; and, lastly, dividing by 20, we find the number of pounds to be 124. (See Ex. 1, page 44.)

```
 4)119040  farthings
   ------
12) 29760  pence
   ------
2,0) 248,0 shillings
   ------
     124  £
   ------
```

2. Reduce 357871 farthings to pounds.

Dividing by 4, as before, we get 89467 pence, and three farthings. Dividing these pence by 12, we get 7455 shillings, and 7 pence; and, lastly, dividing by 20, we get £372. 15*s.*; consequently the entire sum is £372. 15*s.* $7\frac{3}{4}$*d.* (See Ex. 3, page 45.)

$$
\begin{array}{r}
4)357871 \\
\hline
12)\ 89467\tfrac{3}{4} \\
\hline
2,0)\ 745,5 \quad 7d. \\
\hline
£372.\ 15s.\ 7\tfrac{3}{4}d. \\
\hline
\end{array}
$$

* I have been desirous of avoiding purely frivolous questions in this work; but questions such as this must not be considered as such. It is difficult to form an adequate conception of a very large number; and we may be much assisted in doing so by estimating the time it would take to count it. A person will have a much better conception of a *million*, from considering, that if he were to count a million things as fast as possible,—say 100 a minute,—it would occupy him day and night, without intermission, for a whole week, within an hour and 20 minutes.

3. How many days are there in 41961 minutes? (See Ex. 4, p. 45.)

```
6,0)4196,1
   -------
 3) 699    21 m.
    ---
 8) 233
    ---
     29 d.  3 h.  21 m.
    ---
```

Dividing by 60, the number of minutes in an hour, we get 699 hours and 21 minutes. Dividing the hours by 24, that is, by 3 and 8, because $3 \times 8 = 24$, the number of hours in a day, we get 29 days, and 1 *over;* not 1 *hour*, but *once three hours*, because 233, to which this 1 belongs, are not *hours*, but units of 3 hours each; since they arise from dividing hours by 3. And you must be careful to observe this, when in any example in reduction, you split your divisor into its component factors, and divide by *them* instead; the remainder arising from any one of these factors, must always be multiplied by the product of whatever other of those factors may have been previously used for divisors.

4. How many pounds are there in 101157 grains of gold? (See Ex. 5, p. 46.)

```
 3)101157
   ------
 2) 33719
   ------
 4) 16859    3 gr.
   ------
2,0) 421,4  18 gr.
     ---
12) 210     14 dwt.
    ---
     17      6 oz.
    ---
17 lb.  6 oz.  14 dwt.  21 gr.
```

Here we have first to divide by 24, to bring the grains into pennyweights; and the better to illustrate what has just been said, let us split 24 into its *three* factors, 2, 3, and 4, dividing, however, by 3 first, since it is probable, from the last figure of the proposed number being 7, that 3 is contained in it without remainder. The first remainder we get is the remainder 1, from the divisor 2; this remainder multiplied by 3, the only other of the three factors previously used, gives us 3 *grains.* The division by 4 leaves another remainder, namely 3; this multiplied by 6, the product of the 2 and 3, already used as divisors, gives us 18 *grains;* so that the total number of grains over is 21. The final result is, therefore, 17 lb. 6 oz. 14 dwt. 21 gr.

You will observe that I have used the *three* factors of 24 here only for the sake of showing you how to proceed when three factors are necessary in order to enable you to exchange

long division for short division, which you should always do whenever you can.*

Reduce 3275 lb. avoirdupois to cwts.

By the table we see that 112 lb. make one cwt.; it is also easy to see that 112 will divide by 2; the quotient is 56, which is 7×8; therefore, $112 = 2 \times 7 \times 8$; so that we may here use short division, employing these three factors for divisors, as in the margin. Instead of the divisors 2, 7, 8, we might have used 4, 4, 7, as is obvious.†

$$\begin{array}{rl}
2)3275 & \\
\hline
7)1637 & 1 \text{ lb.} \\
\hline
8)\ 233 & 12 \text{ lb.} \\
\hline
29 & 14 \text{ lb.} \\
\hline
29 \text{ cwt.} & 27 \text{ lb.}
\end{array}$$

NOTE. You may sometimes have to divide by a number having a fraction joined to it, as, for instance, by $5\frac{1}{2}$, in order to reduce yards to perches. In this case, the best way is to divide *twice* the dividend by *twice* the divisor, that is, by 11. If the fraction in the divisor be a *quarter* or *three quarters*, instead of a *half*, then you should reduce both dividend and divisor to *quarters* before you begin, multiplying the divisor, without the fraction, by 4, and taking in the odd quarter or quarters. You must observe, however, that the *remainder* you get must be divided by 2, if you are dealing with *halves*, and by 4, if you are dealing with *quarters*, in order that you may obtain the proper remainder, *in the same denomination as the dividend.* Thus, suppose it were required to divide 37810 by $5\frac{1}{2}$, and by $30\frac{1}{4}$, respectively, you should work as follows:—

$$\begin{array}{rr}
 & 37810 \\
 & 2 \\
\hline
5\frac{1}{2} \times 2 = & 11)75620
\end{array}$$

Quotient 6874...3, *half the remainder.*

* A table of factors, suitable for short division, of all numbers up to 10000, is given at the end of the book; the arithmetician will find it very useful on many occasions.

† The learner will readily see that the above method of getting the final remainder corresponds with what has already been explained at page 35. The only difference is, that *there* the remainder is expressed in the *final* denomination, while in Reduction it is made to preserve the *original* denomination. If in the above example the final result were required in cwts. and fractional parts of a cwt. without lbs., the result of the first division would have been written $1637\frac{1}{2}$, the result of the second division $233\frac{13}{14}$, and the result of the third $29\frac{27}{112}$; that is, $29\frac{27}{112}$ cwt., which is 29 cwt. and the 112th part of 27 cwt.; this 112th part being, of course, 27 lb.

$$\begin{array}{ll} & 37810 \\ & 4 \\ 30\tfrac{1}{4}\times 4 & 11)\overline{151240} \\ = 121 = 11\times 11 & 11)\ \underline{13749\ldots 1} \\ & \text{Quotient }\ 1249\ldots 27\tfrac{3}{4},\ \textit{one-fourth of the rem.} \end{array}$$

In the first of these operations, 37810 yards are reduced to poles or perches; in the second, 37810 square yards are reduced to square poles or square perches. The first result is 6874 per. 3 yds.; the second is 1249 sq. per. $27\frac{3}{4}$ sq. yds. You divide the remainder by 2, in the first case, because that remainder is *halves*, and by 4, in the second case, because it is *quarters* or *fourths*.

Exercises.

1. Reduce 26493 farthings to pounds.
2. Reduce 397024 yards to miles, furlongs, perches, and yards.
3. How many hours are there in 28635 seconds?
4. How many pounds of silver are there in 12875 grains?
5. Reduce 176432 lb. to tons.
6. How many yards are there in 24631 nails?
7. Reduce 42657 square poles to acres.
8. How many square yards are there in 27568 square inches?
9. Reduce 100000 pints to gallons.
10. How many degrees and minutes of a circular arc are there in 132530 seconds?
11. How many cubic yards are there in 100000 cubic inches?
12. It is related by Josephus, that the battering-ram employed by Titus against the walls of Jerusalem weighed 100000 lb.: how many tons did it weigh?
13. If an omnibus carry, on the average, 1000 persons weekly at the rate of 3*d.* each, what are the gross receipts for a year?
14. If all the letters which passed through the Post Office during the week ending Feb. 21, 1851, had only penny stamps on them, what was the cost of the stamps? (See Ex. 16, p. 19.)
15. If a steam-vessel sail across the Atlantic Ocean, a distance of about 3000 miles, at the average rate of $9\frac{1}{2}$

miles an hour, in how many days will she perform the voyage?

16. The distance of Plymouth from Adelaide in Australia is estimated at 9080 miles: in how many days would the voyage be performed in a steamer sailing at the average rate of $9\frac{1}{2}$ miles an hour, if no land intervened?

17. The iron railing round St. Paul's Cathedral weighs 448081 lb.: how many tons does it weigh?

18. The duty paid on advertisements in English and Scotch newspapers is 1s. 6d. each: how much was paid for advertisement-duty on all the English newspapers in the year ending Jan. 5, 1851? (See Ex. 14, page 9.)

19. In quick marching, soldiers take 108 steps a minute, each step being about 2 feet 8 inches: at this rate, how long would a regiment be in marching from London to Richmond, a distance of 10 miles?

20. An imperial gallon of distilled water weighs 10 lb. avoirdupois: how many tons of water would the great vat mentioned in Ex. 18, p. 47, hold? (*See foot-note*, p. 42.)

21. A cubic foot of water weighs very nearly 1000 ounces avoirdupois: how many cubic yards are there in the vessel referred to in the last Exercise?

22. In how many days could the above-mentioned vessel be emptied by a tap which discharges half a gallon in a second?

23. What is called a ship-load of coals weighs 949760 lb.; a sack weighs 2 cwt.: how many sacks are there in a ship-load?

24. The number of Electric Telegraph stations now open (Jan. 1, 1852) is 226; of these, constant attendance, day and night, is given at about 70: all are in connection with the central station at Lothbury, in London. The length of lines of communication already completed is upwards of 2500 miles: you can send a short message, for any distance not above 100 miles, for 2*s.* 6*d.*, which message will be forwarded by other means to the house of the person you send to: what would it cost, for the use of the telegraph, to send a message from one end of England to the other, a distance of about 642400 yards,* and to receive an

* The *railway* distance would, of course, be more than this, as the lines do not run directly north and south.

It may interest the learner to be here informed, that the electric wire

answer back, which you might do in a few minutes: you will observe, that 2*s.* 6*d.* is to be paid for *any* distance not exceeding 100 miles?*

(38.) Addition of Compound Quantities.

Rule 1. Place the quantities to be added together under one another, so as that all in the same column may be of the same denomination.

2. Add up the first column on the right, that is, the column in which the quantities of lowest denomination are placed; find how many quantities of the next denomination are contained in the sum: put what is over under the column, and carry the quotient to the next column. Proceed in this way, from column to column, till all have been added up.

1. Suppose, for example, you had to find the total amount of the following bills: namely, Baker's bill, £31 17*s.* 4½*d.*; Butcher's bill, £27 13*s.* 8*d.*; Grocer's bill, £19 0*s.* 6¾*d.*; Tailor's bill, £21 7*s.*; Shoemaker's bill, £11 2*s.* 9*d.*; Washerwoman's bill, £8 16*s.* 3½*d.*; Bookseller's bill, £7 13*s.* 8*d.*; and Stationer's bill, 17*s.* 6½*d.* Then, arranging these sums, as in the margin, putting pounds under pounds, shillings under shillings, pence under pence, and farthings under farthings, you would begin with the column of farthings, and say, 2 and

£.	*s.*	*d.*
31	17	4½
27	13	8
19	0	6¾
21	7	0
11	2	9
8	16	3½
7	13	8
	17	6½
£128	8	10¼

has been extended under the sea, from Dover to Calais; it is embedded in a thick cable, and sunk across the Channel. Occurrences that take place at Paris, 160 miles from Calais, at 7 or 8 o'clock in the evening, are now fully described in print in the London newspapers by 7 o'clock the following morning. Electricity brings the news to London, delivers it in symbols, which require to be translated into common words; the translation is carried in the ordinary way to the printing-office, the compositors set up the type, the pressmen work off the printed sheets, and have thousands of them ready for the public by 7 o'clock in the morning! We owe this wonderful facility to the genius and industry of Professor Wheatstone, of London. The velocity of the electric current is calculated by this gentleman to be at the rate of at least 288000 miles a second; so that it would travel completely round the world in about the twelfth of a second!

* "The most wonderful application of electricity to the purposes of life, is the facility it affords to persons separated by hundreds of miles to

2 are 4, and 3 are 7, and 2 are 9; 9 farthings contain 2 pence, and 1 farthing over; therefore, you put down the 1 farthing, and carry the 2 pence to the column of pence. 2 and 6 are 8, and 8 are 16, and 3 are 19, and 9 are 28, and 6 are 34, and 8 are 42, and 4 are 46; and since 40 pence make 3s. and 4d., 46 pence are 3*s.* and 10*d.*: 10 and carry 3. 3 and 7 are 10, and 3 are 13, and 6 are 19, and 2 are 21, and 7 are 28, and 3 are 31, and 7 are 38; then, proceeding *downwards*, you point to the several *ones* in the *shillings* column, on the left, each 1 standing for *ten*, and say, 48, 58, 68, 78, 88; so that this column amounts to 88 shillings; and since 80 shillings make £4, you put down the 8 shillings, and carry 4 to the column of pounds, the sum of which is 128; so that the total amount of all the bills is £128 8*s.* $10\frac{1}{4}$*d.*

The work of examples in *compound addition* is all so similar to this, that you cannot require any further explanation to prepare you for the following Exercises, which are chiefly intended to give you practice in the tables.

Exercises.

Money.

	£.	*s.*	*d.*		£.	*s.*	*d.*		£.	*s.*	*d.*
1.	13	11	$2\frac{3}{4}$	2.	142	18	0	3.	873	10	$4\frac{1}{4}$
	17	0	$4\frac{1}{2}$		26	9	$7\frac{1}{2}$		327	13	$9\frac{1}{2}$
	0	16	3		14	17	$3\frac{3}{4}$		46	17	2
	2	5	$8\frac{1}{4}$		273	0	$8\frac{1}{4}$		92	8	$10\frac{3}{4}$
	1	17	$5\frac{1}{2}$		97	4	$0\frac{1}{2}$		174	16	$7\frac{1}{4}$
	3	0	0		12	19	$11\frac{3}{4}$		37	9	0
	0	0	$7\frac{3}{4}$		1	6	$5\frac{1}{2}$		18	15	$8\frac{3}{4}$

Time.

	d.	*h.*	*m.*		*d.*	*h.*	*m.*		*d.*	*h.*	*m.*	*s.*
4.	16	13	17	5.	23	19	11	6.	121	14	3	16
	9	2	13		18	6	7		12	9	14	27
	12	17	3		15	17	24		93	21	36	41
	41	9	7		38	11	11		18	19	17	16
	16	21	42		24	23	55		237	12	0	10
	3	15	57		19	13	46		0	23	2	39

hold instant communication, by night or by day, giving them the power, as it were, to annihilate space, enabling them to consult, admonish, inform, condole with each other, as if they were in the same room; and, having ended their conversation, to turn aside, and one to find himself in London, and the other in Edinburgh."—(Sir W. Snow Harris's "Rudimentary Electricity," page 191.)

Avoirdupois.

	lb.	oz.	dr.		cwt.	qr.	lb.		cwt.	qr.	lb.	oz.
7.	10	14	11	8.	5	3	17	9.	23	0	13	14
	17	11	9		17	1	19		14	2	16	13
	21	13	14		32	2	27		45	1	23	11
	6	1	8		1	1	1		9	3	27	8
	12	9	15		14	0	14		19	1	0	15

Troy.

	oz.	dwt.	gr.		oz.	dwt.	gr.		lb.	oz.	dwt.	gr.
10.	7	13	18	11.	9	12	19	12.	13	4	14	20
	5	16	12		10	17	17		6	0	17	2
	11	19	4		0	21	3		0	9	20	1
	8	10	23		8	13	21		25	11	3	23
	10	0	20		7	0	12		16	1	12	18

Apothecaries.

	dr.	scr.	gr.		oz.	dr.	scr.		lb.	oz.	dr.	scr.	gr.
13.	0	2	15	14.	10	6	1	15.	9	11	2	1	14
	7	1	19		5	7	2		7	9	7	2	5
	3	0	17		11	0	0		0	10	3	0	18
	6	2	1		7	1	1		1	7	7	1	19
	4	1	16		3	1	0		13	8	6	0	1

Length.

	yds.	ft.	in.		fur.	po.	yds.	ft.		m.	fur.	po.	yds.
16.	126	1	9	17.	7	14	3	2	18.	124	3	17	2
	37	0	11		6	25	5	1		47	6	20	4
	103	2	8		0	31	4	0		16	1	0	1
	46	1	0		3	19	1	1		230	7	33	5
	234	0	10		5	13	2	2		6	1	2	3

Square Measure.

	ac.	roo.	per.	yds.		ac.	roo.	per.	yds.
19.	127	3	21	13	20.	243	1	18	25
	35	1	17	22		465	2	11	29
	216	2	23	29		43	0	22	17
	13	0	12	17		138	1	15	8
	0	1	8	30		27	2	3	15
	10	3	15	4		0	0	28	6
	1	1	20	18		0	0	36	0

(39.) Subtraction of Compound Quantities.

Rule 1. Place the smaller of the two quantities under the greater, so as that the several parts may be under those of the same denomination.

2. Begin with the *lowest* denomination, and subtract, if the upper number be large enough; if not, increase it by as many as will make *one* of the next higher denomination, taking care, in this case, to carry 1, after the subtraction, to the *next* number you subtract: and proceed in this way till the subtraction is finished.

For example, let it be required to subtract £173 17*s*. $9\frac{1}{2}$*d*. from £241 13*s*. $7\frac{1}{4}$*d*.

£.	s.	d.
241	13	$7\frac{1}{4}$
173	17	$9\frac{1}{2}$
£67	15	$9\frac{3}{4}$

Placing the quantities, as in the margin, and beginning with the lowest denomination, you see that you cannot subtract 2 farthings from 1; you therefore increase the 1 farthing by 4, because 4 farthings make a penny: you then say, 2 from 5, and 3 remain; $\frac{3}{4}$, and carry 1. 1 and 9 are 10, and, increasing the 7*d*., which is too small, by 12*d*., because 12*d*. make a shilling, you say, 10 from 19, and 9 remain; or, it is a trifle easier to say, 10 from 12, and 2 remain, and 7 are 9: carry 1. 1 and 17 are 18; 18 from 20, and 2 remain, and 13 are 15: carry 1. 1 and 3 are 4; 4 from 11, and 7 remain: carry 1. 8 from 14, and 6 remain: carry 1. 2 from 2, and nothing remains: therefore, the difference between the two sums is £67 15*s*. $9\frac{3}{4}$*d*.

Exercises.

	£.	s.	d.		£.	s.	d.		£.	s.	d.
1.	29	11	$4\frac{1}{2}$	2.	465	7	3	3.	2852	0	$7\frac{3}{4}$
	13	16	$8\frac{1}{4}$		258	14	$6\frac{3}{4}$		568	9	$11\frac{1}{2}$

	d.	h.	m.		d.	h.	m.		d.	h.	m.	s.
4.	26	15	17	5.	117	21	43	6.	14	13	5	18
	19	19	19		49	23	57		13	20	32	46

	yds.	ft.	in.		yds.	ft.	in.		per.	yds.	ft.	in.
7.	125	2	11	8.	346	1	7	9.	18	5	2	1
	51	1	6		157	2	10		6	0	1	11

10.	Lat.*	42°	23′	19	N.	11.	Lat.	37°	15′	7″	S.
	Lat.	36	49	25	N.		Lat.	28	38	18	S.

							d.	*h.*	*m.*	*sec.*
12.	Long.	125°	52′	43″	E.	13.	16	21	42	13
	Long.	101	57	56	E.		12	22	58	39

	t.	*cwt.*	*qr.*	*lb.*		*t.*	*cwt.*	*qr.*	*lb.*		*t.*	*cwt.*	*qr.*	*lb.*
14.	7	14	3	19	15.	15	3	1	2	16.	2	0	3	1
	3	18	1	27		9	1	3	17		0	17	0	26

	m.	*fur.*	*per.*	*yd.*		*m.*	*fur.*	*per.*	*yd.*		*m.*	*fur.*	*per.*	*yd.*
17.	128	7	13	2	18.	17	2	18	1	19.	240	0	0	4
	53	6	37	5		1	7	23	4		138	6	12	5

	ac.	*roo.*	*per.*	*yd.*		*ac.*	*roo.*	*per.*	*yd.*		*ac.*	*roo.*	*per.*	*yd.*
20.	73	1	20	6	21.	24	0	14	0	22.	1	2	3	13
	19	2	37	11		17	3	23	31		0	3	9	28

	oz.	*dwt.*	*gr.*		*lb.*	*oz.*	*dwt.*	*gr.*		*lb.*	*oz.*	*dwt.*	*gr.*
23.	13	18	5	24.	9	5	12	12	25.	14	1	3	18
	2	19	23		7	11	17	20		9	0	16	5

	c. yds.	*ft.*	*in.*		*c. yds.*	*ft.*	*in.*		*c. yds.*	*ft.*	*in.*
26.	146	26	271	27.	117	18	110	28.	700	0	0
	107	26	302		53	24	247		384	22	181

	sq. yds.	*ft.*	*in.*		*sq. yds.*	*ft.*	*in.*		*sq. yds.*	*ft.*	*in.*
29.	273	3	17	30.	561	7	110	31.	382	4	46
	187	8	129		359	7	132		75	8	107

	gal.	*qt.*	*pt.*		*gal.*	*qt.*	*pt.*		*gal.*	*qt.*	*pt.*
32.	164	3	0	33.	3492	0	1	34.	4306	1	0
	156	1	1		1783	3	1		3621	2	1

* Ex. 10, is to find the *difference of latitude* of two places on the earth, *north* of the equator; Ex. 11, is to find the difference of latitude of two places *south* of the equator; Ex. 12, is to find the difference of *longitude* of two places *east* of the meridian of Greenwich. The difference of latitude of two places, one *north* and the other *south*, is found by *adding* the two latitudes together; and the difference of longitude of two places, one *east* and the other *west*, is also found by *adding* the two together; what is called the *difference* being, in each case, the interval, in degrees, minutes, and seconds, between the two places.

	bu.	pk.	gal.	qt.		bu.	pk.	gal.	qt.		bu.	pk.	gal.	qt.
35.	18	2	0	3	36.	23	0	0	1	37.	110	1	0	2
	17	1	1	3		17	3	1	3		94	3	1	3

(40.) Multiplication of Compound Quantities.

To multiply a compound quantity by any number, the rule is as follows:—

Rule. Place the multiplier under the quantity of *lowest* denomination. Multiply this quantity by it, divide the product by the *number* of such quantities contained in the *next* denomination, put down the *remainder*, and carry the quotient to the product arising from the next term: and so on to the end.

Note. When the multiplier is greater than 12, and can be decomposed into factors, each not greater than 12, use these factors instead of the composite multiplier, and proceed by *short* multiplication. The table of factors, at the end of the book, will be of great assistance in supplying the proper factors of all composite numbers up to 10000.

Ex. 1. Multiply £17 13*s*. $4\frac{1}{2}$*d*. by 7.

£.	*s*.	*d*.
17	13	$4\frac{1}{2}$
		7
£123	13	$7\frac{1}{2}$

Putting the 7 under the lowest denomination, *farthings*, we multiply the 2 farthings by the 7, the product is 14 farthings; this divided by 4, the number of farthings in a penny, the next denomination, the quotient is 3 *pence*, and 2 *farthings* over; we therefore put down the 2 farthings, namely, $\frac{1}{2}$*d*., and carry 3: we then multiply the 4 pence by the 7; the product is 28, which, with the 3 carried, make 31 pence; dividing these pence by 12, the number of pence in a shilling, the quotient is 2, with 7 pence for remainder; so we put down the 7 pence, and carry the 2 shillings. Multiplying now the 13*s*. by the 7, the product is 91, which, with the 2 carried, make 93 shillings; that is, 4 pounds 13*s*.: 13, and carry 4. And, lastly, multiplying the £17 by 7, and taking in the £4 carried, we have the whole product, £123 13*s*. $7\frac{1}{2}$*d*.

2. Multiply £13 9*s.* $8\frac{3}{4}$*d.* by 693.

£.	s.	d.
13	9	$8\frac{3}{4}$
		11
148	7	$0\frac{1}{4}$
		9
1335	3	$2\frac{1}{4}$
		7
£9346	2	$3\frac{3}{4}$

By looking at the table at the end of the book, we find the multiplier 693 to be a composite number, of which the factors are 11, 9, and 7, we may therefore use these factors as multipliers, and proceed by short multiplication, as in the margin. You cannot require any explanation of the work after attending to the operations in the example just given, so I leave it for you to carefully look over, and thence to form your own opinion of the usefulness of the table of factors in calculations of this kind.

When, however, you have to multiply a compound quantity by a large number, which cannot be decomposed into factors suitable for short multiplication, you may seek in the table for the number nearest to it that *can* be so decomposed; employ the factors of *this* number, and note the result: then, multiply the compound quantity by the *difference* between the given multiplier and that actually used, *add* the result to the former result if the multiplier used be *less* than the given one, and *subtract* if it be *greater*.

There is another way of proceeding, thus: count the number of figures in the multiplier, disregarding the units-figure. Multiply the compound quantity by 10, then the product by 10, and so on, till the number of 10's amount to the same as the number of figures counted: this done, multiply the compound quantity by the units-figure of the given multiplier, the first of the above products by the tens-figure, the next product by the hundreds-figure, and so on, till all the figures of the multiplier have been used: add up all these latter products, and the required product will be obtained. The work of the following example shows both methods.

NOTE.—You must always bear in mind, that a compound quantity can never be *multiplied* by another compound quantity; nor by anything but a mere *number*, since multiplication means the taking a proposed quantity a certain *number of times*. You must at once see the absurdity of the following questions, taken from a recent work on what the author calls "Arithmetic;" namely, "Multiply 7 tons by 9 cwt.;" "Multiply $\frac{3}{8}$ of a £ by $\frac{4}{5}$ of a guinea;" "Multiply $\frac{4}{7}$ of an acre by $\frac{4}{5}$ of a rood;" and so on. As Mr. Walker justly observes (*Philosophy of Arithmetic*, p. 58), "You might as well be told to multiply 5 lbs. of beef by 3 bars of music." I shall have occasion to direct your attention more fully to matters of this kind hereafter.

Ex. Multiply 7*s.* 10$\frac{3}{4}$*d.* by 7985.

By the table, $7986 = 11 \times 11 \times 11 \times 6$.

£.	*s.*	*d.*	
0	7	10$\frac{3}{4}$	$\times 1$
		11	
4	6	10$\frac{1}{4}$	
		11	
47	15	4$\frac{3}{4}$	
		11	
525	9	4$\frac{1}{4}$	
		6	
3152	16	1$\frac{1}{2}$	
Sub.	7	10$\frac{3}{4}$	arising from the multiplier 1, above.
£3152	8	2$\frac{3}{4}$	

£.	*s.*	*d.*		£.	*s.*	*d.*
0	7	10$\frac{3}{4}$	$\times 5 =$	1	19	5$\frac{3}{4}$
		10*				
3	18	11$\frac{1}{2}$	$\times 8 =$	31	11	8
		10				
39	9	7	$\times 9 =$	355	6	3
		10				
394	15	10	$\times 7 =$	2763	10	10
				£3152	8	2$\frac{3}{4}$

I need scarcely tell you, that examples of this kind may always be worked by common reduction; that is, you may reduce the compound quantity to the lowest denomination before multiplying, and then convert the product into the higher denomination.

Exercises.

	£.	*s.*	*d.*			£.	*s.*	*d.*	
1.	32	8	6$\frac{1}{2}$	$\times$ 5	6.	19	13	5$\frac{1}{4}$	$\times$ 28
2.	43	11	4$\frac{3}{4}$	$\times$ 8	7.	21	9	10$\frac{1}{2}$	$\times$ 343
3.	125	13	0$\frac{1}{4}$	$\times$ 12	8.	32	17	1$\frac{3}{4}$	$\times$ 504
4.	217	18	9$\frac{1}{4}$	$\times$ 11	9.	103	11	8$\frac{1}{4}$	$\times$ 891
5.	734	19	7$\frac{3}{4}$	$\times$ 9	10.	379	18	7$\frac{3}{4}$	$\times$ 1617

	miles.	*fur.*	*per.*	*yds.*			*miles.*	*fur.*	*per.*	*yds.*	
11.	15	3	2	4	$\times$ 75	14.	0	6	27	3	$\times$ 594
12.	17	7	0	5	$\times$ 98$\frac{1}{2}$	15.	1	5	19	1	$\times$ 605
13.	23	1	31	2	$\times$ 256	16.	27	3	22	4	$\times$ 972

* In multiplying by 10, you merely annex 0 to the number multiplied; so that when anything below 10 is *carried*, you have only to annex it to the next quantity in the multiplicand.

	days.	*ho.*	*min.*	
17.	19	13	27	$\times 441\frac{1}{4}$
18.	16°	51′	43″	$\times 231\frac{3}{4}$
	tons. cwt.	*qr.*	*lb.*	
19.	14 13	2	11	$\times 243$

	ac.	*ro.*	*po.*	
20.	13	3	17	$\times 511$*
	sq. yds.	*ft.*	*in.*	
21.	2	8	123	$\times 563$*
	oz.	*dwt.*	*gr.*	
22.	9	17	20	$\times 616\frac{1}{4}$

(41.) Division of Compound Quantities.

Rule. Divide the greatest denomination first, put down the quotient; reduce the remainder to the next lower denomination; carry it, thus reduced, to the term of that denomination, in the dividend, and divide as before.

Note 1. When the divisor is a composite number, produced by factors, none of which are greater than 12, use those factors, instead of the number itself, and work by short division.

2. When the divisor is a large number which cannot be decomposed into suitable factors, you may regard the question as one of simple reduction: reduce the compound quantity to the lowest denomination in it, and then divide; the quotient will be a quantity in that lowest denomination, which, by reduction, may then be brought into the higher denomination.

Ex. 1. Divide £34 16*s.* 8*d.* by 24.

	£.	*s.*	*d.*
8)	34	16	8
3)	4	7	1
	£1	9	$0\frac{1}{4}$

Here the divisor is a composite number, formed by the factors 8 and 3. Dividing first by the 8, we say, 8 in 34, 4 times and 2 over: this 2 being *pounds*, we reduce it to *shillings*, carrying the result, 40*s.*, to the 16, and say, 8 in 56, 7 times; then 8 in 8, once. Dividing now by the 3, we say, 3 in 4, once, and 1 over; so that 20*s.* is carried to the 7, and we say, 3 in 27, 9 times; 3 in 1, no times, and 1 over: this 1*d.* being 4 farthings, we say, 3 in 4, once; so that the quotient is £1 9*s.* $0\frac{1}{4}$*d.*, the remainder being neglected, as we have no coin below the farthing.

(42.) Now I have a remark to make upon this operation, to which you must attend. I have supposed that we have

* See page 59.

been working the above example together, and have imagined ourselves as saying, "8 in 34, 4 times;" "8 in 56, 7 times;" and so on, as arithmeticians *would* say: but I must tell you, that we have been using incorrect language; and I am the more anxious to draw your attention to this, in order to show you how necessary it is in reading books of this kind, and, indeed, in reading any book at all, that you should *think for yourself*, and not receive, without thinking, everything that you may find in a printed book. When you had said, as above, "8 in 34, 4 times," and had put down the 4; suppose somebody had asked you what that 4 stood for, you would have answered, 4 *pounds;* and you would have been right: but how can 8 be contained in £34, £4 *times?* 4 pounds *times* is an expression which has no meaning. You see, therefore, that the form of language employed above is faulty: we ought to have said, "the eighth part of £34 is £4, and £2 over;" "the eighth part of 56*s.* is 7*s.*," and so on; or, which is the same thing, "£34 divided by 8 gives £4 for quotient, and £2, or 40*s.*, for remainder;" "56*s.*, divided by 8, gives 7*s.* for quotient, and no remainder;" and so on. You thus see that the only thing that requires correction, in what is done above, is the form of words used in describing the work; but, as the result obtained is always correct, as to the figures, the faulty language, if thought the more convenient, may be allowed to pass; though it is right that you should know what the objection to it is, and how it may be corrected. There is another thing too, to which your notice has been already called. You have seen that multiplication is a short way of finding the result of *addition;* and that *it is nothing more than this:* the multiplier always denotes the *number* of things, each equal to the multiplicand, that are to be added together, and the product gives their sum: the multiplier, therefore, can never be a *commodity*, as a sum of money, or a weight of goods; nor yet any measure of length or space; it is simply what is called an *abstract number*, denoting how many *repetitions* or *times* some other abstract number, or concrete quantity, is to be taken.*

* Among those who have advanced further into the practical application of arithmetic, there may be some who may think that this view of multiplication is in opposition to what goes by that name in books on mensuration, surveying, &c., where feet are apparently multiplied by feet, yards by yards, &c. The fact is, however, that although concrete quantities are in such subjects *said* to be multiplied together, the phrase-

Now, just as multiplication is an abridgment of addition, so it has been said that division is an abridgment of repeated subtraction; so it is: but it is also more than this: we could not have any operation in multiplication which could not be performed by addition, though there are plenty of operations in division which could not be performed by subtraction: how, for instance, could the example worked above be done by subtraction? A sum of money, which is a *concrete* quantity, a real commodity, is to be divided by 24, an *abstract number*, not 24 *things;* it would be nonsense to speak of taking an abstract number from a concrete quantity,—from real substantial things: when we divide a concrete quantity by 24, we merely seek that smaller concrete quantity which is the 24th part of the greater; or that smaller quantity, which repeated 24 times, makes up the greater. Division of a concrete quantity replaces subtraction, only when the divisor is a concrete quantity of the same kind *also*. A sum of money may, for instance, be divided by a smaller sum of

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ology is adopted solely for brevity, and to enable writers on those topics to express the rules of operation in a form easy of recollection, and free from that prolixity of language which the strictly correct form of expression would seem to require. All that is meant is, that we are to proceed in applying the rules of mensuration, &c., *as if* feet could be multiplied by feet, yards by yards, &c., or *as if*, instead of these concrete quantities, they were merely abstract numbers. The direction for finding the surface of a rectangle is briefly expressed by saying, "multiply the length by the breadth, the product will be the area in *square* feet, &c." The meaning is, that we are to multiply these measures together *as if* they were not measures, but abstract numbers; and then to consider the product *as if* it were not an abstract number, but so many *square* feet, &c.; this is all that is to be understood by the expression "feet multiplied by feet produces square feet;" and the same of all expressions of a like kind. Strange to say, however, there are books on arithmetic,—and books, too, of very recent date,—the authors of which, teaching the subject as they themselves have learnt it, that is, merely as a sort of mechanical jugglery with the nine digits—I say, there are modern books on arithmetic, the authors of which, finding the expression "feet multiplied by feet" tolerated, proceed to induct their deluded pupils into the mystery of multiplying cwts. by tons, money by money, and so on! What meaning they attach to their results no one knows; indeed, *meaning*, or any accounting for their processes by an appeal to reason or common sense, is what never enters the heads of these writers; they would, no doubt, just as readily multiply a house by a house, or one man's name by another's. There is perhaps no class of educational books which has done so much injury to the youthful mind as books on arithmetic. What a benefit would it be to the young if about five-sixths of existing works on arithmetic were collected in one vast pile, and burnt in Smithfield for scientific heterodoxy!

money: in this kind of division, we merely seek *how many times* the smaller sum is contained in the greater; the quotient must evidently, therefore, be an abstract number. It is only when dividend and divisor are of the same kind, both concrete quantities of one sort, or both abstract numbers, that the operation of division can replace that of successive subtraction.

(43.) I have directed your attention to these particulars, in order that you may clearly see the true character of your operations in the multiplication and division of compound quantities, and to impress upon you, that a multiplier is always an *abstract number*, while a divisor may be either an abstract number, or a *concrete quantity of the same kind as the dividend;* and, moreover, to prepare you for the rule for division in this latter case: this rule is as follows:—

(44.) *To divide a Compound Quantity by Another of the Same Kind.*

RULE. Reduce the two quantities to the *lowest* denomination to be found in *either*, and then perform the division: the quotient will express the number of times the smaller quantity is contained in the greater.

Ex. Divide £18 5*s.* by £2 7*s.* 8*d.*

Here the lowest denomination is *pence;* we have, therefore, to reduce both dividend and divisor to this denomination, and then to divide the pence contained in the dividend by the pence contained in the divisor, as in the margin. We thus find that the smaller sum is contained in the greater 7 times and a part of a time, expressed by the fraction $\frac{376}{572}$, which part is the 572nd part of the number 376. If the sum to be divided were diminished by 376 pence, then the other sum would be contained in it exactly 7 times.

£.	*s.*	*d.*		£.	*s*
2	7	8		18	5
20				20	
47				365	
12				12	
572			)	4380($7\frac{376}{572}$	
				4004	
				376 rem.	

Exercises in both Rules.

1.	£148 16*s.* 4*d.* ÷ 8	4.	£106 19*s.* $3\frac{1}{4}$*d.* ÷ 72
2.	237 13 5 ÷ 14	5.	780 12 $9\frac{1}{2}$ ÷ 168
3.	562 18 $6\frac{1}{2}$ ÷ 35	6.	837 13 $5\frac{1}{2}$ ÷ 273

7. 14 cwt. 1 qr. 9 lb. ÷ 18
8. 823 m. 7 fur. 21 po. ÷ 11
9. 128° 45′ 52″ ÷ 125
10. 315 d. 17 h. 38 m. ÷ 112
11. 1784 ac. 3 roo. 32 per. ÷ 105.

12. £15 16*s.* 9*d.* ÷ £2 13*s.* 5*d.*
13. £89 11*s.* 7½*d.* ÷ £7 3*s.* 4¼*d.*
14. £126 7*s.* ÷ £34 18*s.* 1¾*d.*
15. £321 17*s.* 3¾*d.* ÷ £47 6*s.* 9¾*d.*
16. 73 cwt. 3 qr. 13 lb. ÷ 5 cwt. 1 qr. 14 lb.
17. 78 d. 18 h. 49 m. ÷ 5 d. 1 h. 2 m.
18. 79° 13′ 46″ ÷ 13° 5′ 18″
19. 2 tons 13 cwt. 5 lb. ÷ 3 qr. 17 lb.
20. The mint price of standard gold is £3 17*s.* 10½*d.* an ounce: what is the value of 1 lb.?
21.* The amount of money expended for the maintenance of the poor among the 607 Unions of England and Wales for the year ending at Michaelmas, 1851, was £3288192: how much, on the average, was expended by each Union?
22. A fruiterer offers a market-woman 120 oranges at 3 a penny, and 120 of a better sort at 2 a penny, and refuses to take any less; the woman offers to purchase the whole at 5 for 2*d.*; and the man, thinking that this is the same thing, lets her have them: how much did the woman *save* by this arrangement?
23. The building of the new Royal Exchange in London cost £400000; and after it was opened by the Queen on Monday, the 28th of October, 1844, the public were admitted to it for three days, when a subscription was made for the widows of four men killed during the progress of the works: the money received was as follows: 4 sovereigns, 1 half-sovereign, 1 crown piece, 88 half-crowns, 992 shillings, 842 sixpences, 142 four-penny pieces, 5 threepenny pieces, 665 pence, 667 half-pence, and 25 farthings: what was the total amount, and what was each widow's share?
24. If a dozen teaspoons weigh 9 oz. 18 dwt. 20 gr., what is the weight of each spoon?
25. Although a sovereign, when quite new, weighs very

* These questions are of a miscellaneous kind; some of them require only the rules for reduction. This mixed character is intentionally given to them, that the learner may be accustomed to work examples without requiring to know what *rule* they come under.

nearly $123\frac{2}{7}$ grains, yet there are only 113 grains of pure gold in it, the rest is called *alloy*, and is either of pure copper, or a compound of silver and copper:* for how many sovereigns was there gold sufficient in the 312500 ounces supposed to have been collected in California during the year 1850?

26. Find the value of the mass of gold in last Example, at the rate of only £3 17*s*. $10\frac{1}{2}$*d*. an ounce. NOTE. $312500 = 10 \times 10 \times 5 \times 5 \times 5 \times 5 \times 5$: see table of factors at the end.

27. If 64 lb. of tea, at 4*s*. 8*d*. a pound, be mixed with 42 lb. at 4*s*. 4*d*. a pound, what per lb. will be the price of the mixture?

28. How much would the money which the Royal Exchange cost weigh in *sovereigns*, at 123 grains each;† and how many times as high as the Monument (202 feet) would they reach, if piled one on another, allowing a pile of 16 to reach an inch, which is about the case?‡

29. A bankrupt owes his creditors £2831, and proposes to pay them 13*s*. $2\frac{1}{2}$*d*. in the pound: how much money must he have to do this?

30. What is the difference in the weight of 100000 sovereigns and 100000 guineas; the weight of a sovereign being $123\frac{1}{4}$ grains, and the weight of a guinea $129\frac{1}{2}$ grains?

31. From the 31st of December, 1829, to the 15th of February, 1831, there were coined at the Mint 2387881 sovereigns: what was the weight of these, and what weight of pure gold was used in the coinage?§ (See Ex. 25.)

* The pound troy, of sterling gold, contains 11 oz. of pure gold, and 1 oz. of alloy; it is coined into $46\frac{29}{40}$ sovereigns.

† That is, supposing each sovereign had lost, on the average, a quarter of a grain by use. It may be here mentioned, that gold coins are allowed to pass under the mint weight, in consideration of the effects of wear. A sovereign weighing $122\frac{3}{4}$ grains, is considered a legal tender; but not if it be below this weight.

‡ Of sovereigns much worn by use, about 17 would be required to make an inch: of new sovereigns, 16 would do.

§ In expressing the weight of gold, it is not usual to employ higher denominations than *pounds*; when cwts. and qrs. are mentioned in troy weight, cwt. means 100 lb., and qr. means 25 lb.

FRACTIONS.

(45.) I AM now about to explain to you one of the most important parts of arithmetic,—the arithmetic of *fractions*. Learners generally consider it to be the most *difficult* part; but I am sure that if you carefully attend to the explanations to be given, you will find the arithmetic of *fractions* to be quite as easy as the arithmetic of *integers*. I have been obliged to mention the term *fractions* already, and to say a little about them; for you see they *will* force themselves upon our notice at a very early stage of arithmetic. I am now to speak of them more at length, and must begin by *defining* a fraction; that is, by telling you what a fraction really is.

A fraction, strictly speaking, means a part of *unit*, or 1; thus, *one-half*, *two-thirds*, *three-fourths*, &c., are *fractions;* they are represented by the figures of arithmetic, in the following way, namely, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, &c. These are so many examples of the *notation* of fractions; the number below the short line is always called the *denominator*, and the number above it the *numerator;* because the lower number always makes known to us the *denomination* of the parts, as to whether they are *halves*, or *thirds*, or *fourths*, &c., and the upper number tells us how many of those parts are meant; that is, it *enumerates* them. If therefore a fraction such as $\frac{5}{7}$ were presented to you, you would at once know what was meant by it. Looking at the *numerator*, you would see that 5 *parts* were represented, and looking at the denominator to learn *what* those parts were, you would see that they were *sevenths;* you would thus know that the fraction, translated into words, is *five-sevenths;* that is, if unit, or 1, were cut up into seven equal parts, five of those parts are represented by $\frac{5}{7}$.

You must see that it is very convenient to have a *notation* for parts of a unit as well as for whole units or *integers;* and from what has already been said, you also see that, in strictness, a fraction is always something *less* than 1. But this strictness is departed from; the term *fraction* is extended to quantities *greater* than 1: thus, $\frac{7}{5}$, $\frac{8}{3}$, $\frac{11}{4}$, &c., are all called *fractions*, though each is greater than 1. The first stands for *seven-fifths*, the second for *eight-thirds*, the third for *eleven-fourths*, and so on. The *meaning* of the first is, that if 1 be

cut up into *five* equal parts, seven of *such* parts, that is seven times *one* of those parts, are to be taken; the meaning of the second is, that if 1 be divided into *three* equal parts, eight times *one* of those parts is to be taken; and the third fraction means, that if 1 be divided into *four* equal parts, eleven times *one* of them is to be taken. Fractions such as these, where each denotes a quantity *greater* than unity, are called *improper fractions*,—the prefix *improper* reminding us that the strict meaning of the word *fraction* is departed from, a *proper* fraction always having the numerator *less* than the denominator. Even when the numerator is *equal* to the denominator, the thing is still called a fraction,—an *improper* fraction, of course; thus, $\frac{3}{3}$, $\frac{4}{4}$, $\frac{5}{5}$, &c., are all called fractions, though in fact each is only a peculiar manner of writing down unit, or 1, since three-thirds, four-fourths, &c., each make one *whole*. There is thus no restriction upon fractional notation; you may write any number you please for numerator, and any number you please for denominator, and what you put down will be entitled to be called a *fraction*.

(46.) I think from what has now been said, you will see that fractions are a good deal like those quantities with which you have been occupied in the preceding articles, where both the *number* and the *denomination* of the things dealt with are to be considered, the chief difference being merely in the *notation*. When the things dealt with were pounds, in money, the denomination was expressed by the mark, or symbol, £, written against the *number* of pounds; when the denomination was ounces, the symbol oz. was employed in the same way; and so on. In like manner in fractions, both number and denomination have to be expressed; but here a different notation is used,—the *number* of the things being written *above*, and their *denomination* below a short line of separation; so that when you are familiar with the notation, you ought to find no more difficulty in the arithmetic of *fractions* than in the arithmetic of *whole* quantities of different denominations.

(47.) You have already been told (p. 28), that this notation for fractions is also the notation for *division;* so that $\frac{2}{3}$ ought to mean not only *two-thirds* of unit, or 1, but also 2 *divided by* 3; and that $\frac{3}{4}$ should express, indifferently, either *three-fourths*, or 3 *divided by* 4; and so on. And it is pretty plain that such is really the case; for *two-thirds*, or the third part of 1 taken *twice*, is obviously the same as the third part

of 2 taken *once;* that is, 2 divided by 3: also, that *three-fourths*, or the fourth part of 1 taken 3 *times*, is the same as the fourth part of 3 taken *once;* that is, 3 divided by 4, and so on. Consequently, whenever you see a fraction, as $\frac{5}{7}$, $\frac{9}{4}$, &c., you may read it either *five-sevenths*, *nine-fourths*, &c., or 5 *divided by* 7, 9 *divided by* 4, &c. Suppose, for instance, you had $\frac{3}{4}$ (that is, *three-fourths*) of 1 *shilling*, or 12 *pence;* then, since *one* fourth is 3*d*., three fourths would be 9*d*., which you see is one fourth of 3*s*., or of 36 pence; that is, 36 pence divided by 4 gives 9*d*. Again, $\frac{9}{4}$ (that is, *nine-fourths*) of a shilling is 9 times one-fourth, or 9 times 3*d*., which is 27*d*., and 9*s*., or 108 pence, divided by 4, is also 27*d*.; and similarly in all cases: and it is of importance that you keep this fact always in remembrance.

(48.) Before proceeding to the arithmetic of fractions, I have only further to add, that

1. A whole number, that is an *integer*, may be written in the form of a fraction, by merely putting under it 1 for denominator; thus, 3 may be written $\frac{3}{1}$; 7 may be written $\frac{7}{1}$, and so on. And that *any* number may be written for denominator, provided only the product of that number and the proposed one be written for numerator; thus, if we wish to write 3 in a fractional form, with 5 for denominator, we must write 3×5, or 15, for numerator, the fraction being $\frac{15}{5}$, which is, of course, the same as 3. In like manner, $7 = \frac{35}{5}$, or $= \frac{42}{6}$, or $= \frac{49}{7}$, &c. The numerator and denominator of any fraction are called the *terms* of the fraction.

2. A number consisting of two parts, one *whole* and the other *fractional*, is called a *mixed* number: thus, $2\frac{1}{2}$, $3\frac{5}{7}$, $22\frac{3}{11}$, &c. are all mixed numbers. Such mixed numbers may always be reduced to *improper* fractions; and, on the other hand, an improper fraction may always be reduced to a mixed number. It will be as well to commence the subject by showing how these reductions are to be made.

(49.) *To reduce a Mixed Number to an Improper Fraction.*

RULE. Multiply the whole number by the denominator of the fraction connected with it. Add the product to the numerator, and write the denominator of the fraction underneath, with the short line of separation between, and you will have the improper fraction required.

Thus, if $3\frac{5}{7}$ be the mixed number, consisting of the integer

3 and the fraction $\frac{5}{7}$, we should say, 3 times 7 are 21, which added to the numerator 5 make 26, which is therefore the numerator of the equivalent improper fraction, the denominator being the same as that in the given fraction: therefore $3\frac{5}{7}=\frac{26}{7}$. In like manner, $4\frac{3}{8}=\frac{35}{8}$; $16\frac{4}{5}=\frac{52}{3}$; $13\frac{11}{9}=\frac{128}{9}$; and so on. And this merely amounts to writing the given whole number in a fractional form with the given denominator for *its* denominator; thus, taking the last instance above, namely $13\frac{11}{9}$, the 13 is the same as 9 times 13 divided by 9; that is, $13=\frac{117}{9}$; therefore 13 *and* $\frac{11}{9}$, or, as it is written, $13\frac{11}{9}=\frac{128}{9}$. The reason of the rule is thus evident.

(50.) *To reduce an Improper Fraction to a Mixed Number.*

Rule. Actually perform the division denoted by the fraction, and to the quotient annex the *remainder* with the divisor underneath; that is, complete the quotient by adding the fractional correction. Thus, performing the division implied in $\frac{26}{7}$, the *integral* part of the quotient is 3, with 5 for remainder; so that the *fractional* part of the quotient is $\frac{5}{7}$: therefore the complete quotient is the mixed number $3\frac{5}{7}$.

These two rules are so easy and obvious, that but a very few exercises in them need be given.

Exercises.

1. Reduce $7\frac{2}{3}$ to an improper fraction.
2. Reduce $17\frac{3}{7}$ to an improper fraction.
3. Reduce $\frac{139}{12}$ to a mixed number.
4. Reduce $\frac{2075}{17}$ to a mixed number.
5. Reduce $238\frac{19}{27}$ to an improper fraction.
6. Reduce $\frac{31872}{123}$ to a mixed number.
7. Reduce $2016\frac{27}{10}$ to an improper fraction.
8. Reduce $\frac{12731}{317}$ to a mixed number.

(51.) *To reduce Fractions with Different Denominators to others with Equal Denominators.*

This reduction is called the reduction of fractions to a *common denominator*: it is a change necessary to be made in fractions of *different* denominators, before they can be

either added together, or subtracted one from another. You will be prepared to expect this; for you know that things of one denomination cannot be added to or subtracted from things of a different denomination, till they are prepared in this way: shillings and pence, or cwts. and lbs., cannot be united together in one result, without distinction of denomination, unless the differing denominations be first changed to *common* denominations, that is, to denominations the same in, or common to, *both*. In like manner, $\frac{2}{3}$ cannot be either added to or subtracted from $\frac{4}{5}$, till the fractions are changed into others, equal to them in value, and of a common denominator; since *thirds* and *fifths*, being different denominations, cannot be united together in a single denomination. The desired change may always be effected by aid of the following principle; namely, *If both terms of a fraction be multiplied by any number, the two products may be put for the terms themselves;* and this is only saying, that we may multiply a dividend and its divisor by any number we please, without altering the quotient: and it is plain, that the quotient of a dividend by its divisor, is the same as the quotient of *twice* the dividend by *twice* the divisor; *three* times the dividend by *three* times the divisor; and so on, for any number of times: if, for instance, a sum of money is to be divided among a certain number of people, the share of each must be the same as if *twice* the sum were to be divided among *twice* the number of people, or as if 8 times the sum were to be divided among 8 times the number; 10 times the sum among 10 times the number; and so on. The liberty thus given to us, to multiply the terms of a fraction by any number we please, enables us to change those fractions having *different* denominators into others, equal to them in value, but with the *same* denominators, by the following rule:—

(52.) RULE 1. Multiply the numerator of *each* fraction by the product of the denominators of all the *other* fractions: the several results will be the several numerators of the changed fractions.

2. Multiply *all* the denominators together: the product will be the denominator common to all.

Thus, to change the fractions $\frac{1}{2}$, $\frac{3}{5}$, $\frac{7}{9}$, into others of the same value, and with a denominator common to all, we multiply the numerator 1, of the first, by 45, the product of the denominators 5 and 9 of the *other* fractions; we then multiply the numerator 3, of the next fraction, by 18, the product

of the denominators 2 and 9 of the *other* fractions; and, lastly, we multiply the numerator 7, of the next fraction, by 10, the product of the denominators 2 and 5 of the *other* fractions: we thus get for the *numerators* of the new fractions, 45, 54, and 70; and for the *common denominator*, $2 \times 5 \times 9 = 90$. Hence the proposed fractions, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{7}{9}$, are, respectively, equal to $\frac{45}{90}$, $\frac{54}{90}$, $\frac{70}{90}$: for these are no other than the former fractions, after numerator and denominator of each are both multiplied by the *same number*: both terms of the first fraction, $\frac{1}{2}$, are multiplied by 45; both terms of the second, $\frac{3}{5}$, by 18; and both terms of the third, $\frac{7}{9}$, by 10. And similarly, in all cases, by following the directions of the rule, we multiply the terms of *each* fraction by the product of the denominators of all the *other* fractions; so that though the fractions are changed in *appearance*, they remain unchanged in *value*.

Exercises.

Reduce the following fractions to others of equal values, having a common denominator.

1.	$\frac{1}{2}$,	$\frac{2}{3}$,	$\frac{1}{5}$	7.	$\frac{3}{4}$,	$\frac{2}{5}$,	$\frac{6}{7}$,	$\frac{1}{8}$
2.	$\frac{3}{4}$,	$\frac{2}{5}$,	$\frac{1}{3}$	8.	$\frac{4}{5}$,	$\frac{3}{8}$,	$\frac{5}{6}$,	$\frac{1}{2}$
3.	$\frac{3}{5}$,	$\frac{2}{7}$,	$\frac{4}{9}$	9.	$\frac{6}{7}$,	$\frac{3}{5}$,	$\frac{4}{3}$,	$\frac{3}{4}$
4.	$\frac{3}{8}$,	$\frac{5}{6}$,	$\frac{7}{3}$	10.	$\frac{5}{9}$,	$\frac{3}{11}$,	$\frac{2}{7}$,	$\frac{1}{4}$
5.	$\frac{7}{9}$,	$\frac{3}{11}$,	$\frac{5}{7}$	11.	$\frac{1}{2}$,	$\frac{3}{7}$	$\frac{7}{2}$,	$\frac{5}{9}$
6.	$\frac{2}{13}$,	$\frac{9}{10}$,	$\frac{11}{12}$	12.	$\frac{3}{4}$,	$\frac{3}{5}$,	$\frac{1}{7}$,	$\frac{5}{9}$

(53.) You see, from these examples, that the rule just given will always enable you to convert a set of fractions, with different denominators, into another set equal to them in value, with the same denominators. Any rule would do that would always supply us with a set of multipliers, for the terms of the several fractions, such that the products derived from the *denominators* should be all alike: the smaller such suitable multipliers are, the neater and simpler will be the changed forms; and such smaller multipliers often suggest themselves, by merely passing the eye along the row of denominators; for instance, if the original fractions were $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{8}$, you would see in a moment, that the denominators would give equal products if the first were multiplied by 4, the second by 2, and the third left as it is, without any multiplication at all: therefore, multiplying both terms of the first fraction by 4,

both terms of the second by 2, and leaving the third untouched, we have the changed forms, $\frac{4}{8}$, $\frac{6}{8}$, $\frac{5}{8}$:—three fractions equivalent to the original ones, and all of the same denomination—*eighths*. By the *rule*, the given fractions would have been changed into $\frac{32}{64}$, $\frac{48}{64}$, $\frac{40}{64}$, which are less simple in appearance than the other set, though the same in value: the former would be converted into these by multiplying both terms of each by 8. You thus perceive, that before applying the general rule to a set of fractions, it will always be prudent to look a little at the row of denominators, and try to find out, whether smaller multipliers than those which the rule would give you cannot be discovered: the smallest possible always *can* be discovered by a mode of proceeding which I will show you presently: but the fewer the *rules* you depend upon the better; a little thought and attention will often supply their place. I shall therefore give you a few fractions to be reduced to a common denominator, without appealing to the rule; first, however, noticing that, as the terms of a fraction may be *multiplied* by any number, so they may be *divided* by any number, whenever such division is possible: thus, $\frac{4}{8}$ is reducible to $\frac{1}{2}$; $\frac{6}{14}$ to $\frac{3}{7}$; and so on, as is plain, because by *multiplying* both terms, we know that $\frac{1}{2} = \frac{4}{8}$, $\frac{3}{7} = \frac{6}{14}$, &c. It would be considered as an *arithmetical fault* in a person, who pretended to a knowledge of fractions, to leave a fraction at the close of his work, of which the numerator and denominator have an *obvious common divisor*. Be careful to avoid this fault: never allow your work to end with such a form as $\frac{6}{21}$, or $\frac{8}{12}$, or $\frac{15}{27}$, &c., where the simplifying divisors are *obvious:* the final forms in which these fractions should be put, are $\frac{2}{7}$, $\frac{2}{3}$, and $\frac{5}{9}$, which are incapable of further simplification. In the fractions which form part of the complete quotients, in the exercises on Division (p. 35), instances occur, where what would *now* seem an obvious simplification, is neglected; but you could not be supposed to know *then*, that the terms of a fraction might be divided by a number without changing its value.

It may be of use to you to know and to remember, that a number is divisible by 2, if its last figure be either an even number or 0; that it is divisible by 3, if the sum of its digits (or figures) be divisible by 3; by 4, if the number expressed by its last *two* figures be divisible by 4; by 8, if the number denoted by its last *three* figures be divisible by 8; by 5, if its last figure be either 5 or 0; and by 9, if the sum of its digits be divisible by 9.

(54.) To give you some guidance in this search, I will show you how to proceed with Example 21, below. You observe here, that the denominators 4 and 6, of the first and third fractions, have a common factor, 2; the one denominator being 2×2, and the other 3×2. Now these denominators will become equal if the factor 3 be introduced into the *first*, and the factor 2 into the *second;* for then each will be composed of the same factors, 2, 2, and 3; therefore, multiplying the terms of the first fraction by 3, and the terms of the second by 2, the fractions become $\frac{3}{12}$, $\frac{2}{12}$; the other two fractions are $\frac{2}{10}$, $\frac{3}{10}$. The only differing denominators are now 12 and 10; these have a common factor, 2; for they are 6×2, and 5×2; they are therefore made equal by multiplying the first by 5, and the second by 6. Hence, as before, using these multipliers for the fractions last deduced, they become $\frac{15}{60}$, $\frac{10}{60}$, $\frac{12}{60}$, $\frac{18}{60}$.

13.	$\frac{3}{4}$,	$\frac{5}{8}$,	$\frac{4}{16}$	19.	$\frac{6}{27}$,	$\frac{8}{9}$,	$\frac{7}{18}$,	$\frac{1}{3}$
14.	$\frac{2}{3}$,	$\frac{3}{5}$,	$\frac{7}{9}$	20.	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{1}{4}$,	$\frac{1}{5}$
15.	$\frac{5}{7}$,	$\frac{12}{9}$,	$\frac{13}{24}$	21.	$\frac{1}{4}$,	$\frac{1}{5}$,	$\frac{1}{6}$,	$\frac{3}{10}$
16.	$\frac{2}{6}$,	$\frac{7}{21}$,	$\frac{8}{27}$	22.	$\frac{8}{36}$,	$\frac{9}{27}$,	$\frac{5}{12}$,	$\frac{7}{9}$
17.	$\frac{5}{108}$,	$\frac{2}{9}$,	$\frac{11}{54}$	23.	$2\frac{1}{6}$,	$\frac{5}{12}$,	$\frac{2}{5}$,	$\frac{1}{4}$
18.	$\frac{7}{8}$,	$\frac{4}{25}$,	$\frac{8}{100}$	24.	$3\frac{1}{5}$,	$\frac{3}{8}$,	$\frac{3}{25}$,	$\frac{4}{15}$

The reduction of fractions to others with a common denominator, besides being a needful preparation for adding and subtracting, is often also necessary to enable us to see which of two fractions is the greater: thus, $\frac{2}{7}$ and $\frac{3}{11}$ differ so little, that till you find, by multiplying each numerator by the denominator of the other fraction, that 2×11 exceeds 3×7, you would not know that the first is the greater.

(55.) Addition of Fractions.

Rule. Reduce the fractions to others having a common denominator. Add the numerators of the latter together,

* I need scarcely say, that when fractions are changed to others with equal denominators, it is not possible to preserve this equality, and at the same time to express each fraction in its lowest terms: as the terms of one or more of the given fractions have to be multiplied by some number to bring about the equality of the denominators, *both* objects cannot, of course, be accomplished.

and under the result write the common denominator; the fraction thus formed will be the sum of those proposed.

If any mixed numbers are to be added, the *fractional* portions only of such numbers are to be added by the preceding rule, and the sum of the integral parts to be added afterwards.

NOTE. Those fractions, of which the denominators have a factor in common, you will find it much more convenient to add two at a time, as in the examples 2, 3, and 4, below.

Ex. 1. Add together the fractions $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{5}{7}$.

These reduced to a common denominator are $\frac{56}{84}$, $\frac{63}{84}$, and $\frac{60}{84}$; and $\frac{56}{84}+\frac{63}{84}+\frac{60}{84}=\frac{179}{84}=2\frac{11}{84}$, the sum.

2. Required the sum of $2\frac{3}{5}$, $1\frac{2}{7}$, and $\frac{9}{10}$.

Here, taking the fractions only, we see at once that the first and third are $\frac{6}{10}$ and $\frac{9}{10}$, the sum of which is $\frac{15}{10}=1\frac{1}{2}$; therefore, taking in the second fraction, we have $\frac{1}{2}+\frac{2}{7}=\frac{7}{14}+\frac{4}{14}=\frac{11}{14}$; and since the sum of the whole numbers is $2+1+1=4$, the sum of the proposed quantities is $4\frac{11}{14}$.

3. Add together $3\frac{31}{40}$, $\frac{29}{36}$, and $6\frac{23}{25}$.

Here it is at once seen that the denominators 40 and 36 each have a common factor, 4; the first two fractions may therefore be written $\frac{31}{10\times4}$, $\frac{29}{9\times4}$; so that the two denominators will be the same, if the first be multiplied by 9 and the second by 10; for the factors of each will then be 4, 9, and 10. Consequently, $\frac{31}{40}+\frac{29}{36}=\frac{279}{360}+\frac{290}{360}=\frac{569}{360}=1\frac{209}{360}$. The third fraction, after multiplying its terms by 2, is $\frac{46}{50}$; and $\frac{209}{360}+\frac{46}{50}=\frac{1045}{1800}+\frac{1656}{1800}=1\frac{901}{1800}$; hence the sum is $11\frac{901}{1800}$. The reason why I changed $\frac{23}{25}$ into $\frac{46}{50}$ was, that 50 is a more convenient number to multiply by than 25.

4. Add together $\frac{3}{16}$, $\frac{5}{12}$, $\frac{4}{15}$, and $\frac{19}{40}$.

The denominators of the first two fractions are 4×4 and 3×4; these are made alike by the factor 3 for the first, and 4 for the second; using therefore these for multipliers, the first two fractions become changed into $\frac{9}{48}$, $\frac{20}{48}$. The denominators of the other two fractions are 5×3 and 5×8; they are made alike by the factor 8 for the first, and 3 for the second. Hence the last two fractions are changed into $\frac{32}{120}$, $\frac{57}{120}$. The only differing denominators are now 48 and 120; that is, 24×2 and 24×5, which are made alike by the factor 5 for the first, and 2 for the second; therefore, using these for multipliers of $\frac{29}{48}$ and $\frac{89}{120}$, the sums of the pairs of fractions above, we have, finally, $\frac{145}{240}+\frac{178}{240}=\frac{323}{240}=1\frac{83}{240}$.

And this is the way in which you should proceed in working the following exercises, reducing the fractions to a common denominator, *by the rule*, only when no two of the denominators have a factor in common. If you are only careful always to select the *greatest* factor in common, your changed fractions will always have the *least* common denominator. A rule will presently be given for enabling you to find the greatest common factor of two numbers, however large those numbers may be.

Exercises.

1. $\frac{1}{4}+\frac{2}{5}+\frac{3}{7}$.
2. $\frac{3}{8}+\frac{1}{6}+\frac{4}{5}$.
3. $\frac{5}{12}+\frac{3}{10}+1\frac{1}{7}$.
4. $\frac{4}{15}+\frac{5}{8}+6$.
5. $2\frac{7}{9}+3\frac{2}{3}+\frac{1}{2}$.
6. $8\frac{1}{2}+5\frac{11}{12}+\frac{3}{8}$.
7. $\frac{15}{3}+\frac{7}{24}+3\frac{5}{9}$.
8. $\frac{2}{27}+\frac{7}{54}+1\frac{2}{5}$.
9. $2\frac{17}{25}+1\frac{3}{10}+\frac{13}{16}$.
10. $1\frac{9}{14}+\frac{11}{63}+\frac{2}{15}+3\frac{7}{20}$.
11. $\frac{14}{15}+\frac{11}{12}+\frac{26}{27}+\frac{39}{40}$.
12. $\frac{35}{27}+\frac{3}{18}+5\frac{9}{28}+\frac{5}{12}$.
13. $\frac{19}{63}+\frac{17}{36}+\frac{11}{26}+\frac{5}{39}$.
14. $6\frac{13}{35}+1\frac{7}{27}+\frac{8}{11}+\frac{1}{4}$.
15. $\frac{5}{18}+\frac{7}{27}+\frac{19}{28}+4\frac{9}{21}$.
16. $2\frac{8}{13}+1\frac{2}{11}+3\frac{9}{15}+\frac{5}{7}$.

(56.) Subtraction of Fractions.

Rule. Reduce the fractions to a common denominator, which place under the difference of the changed numerators.

As in addition, after the fractions are brought to a common denominator, the *numerators only* are *added*, so in subtraction, the numerators *only* are subtracted; the common denominator in both operations being preserved, as it is merely that which denotes *what* the things added or subtracted are.

Ex. 1. Subtract $\frac{5}{7}$ from $\frac{4}{5}$. Here $\frac{4}{5}-\frac{5}{7}=\frac{28}{35}-\frac{25}{35}=\frac{3}{35}$.

2. Subtract $2\frac{3}{8}$ from $4\frac{7}{12}$.

Here $4\frac{7}{12}-2\frac{3}{8}=4\frac{14}{24}-2\frac{9}{24}=2\frac{5}{24}$.

3. Subtract $3\frac{3}{4}$ from $7\frac{2}{3}$.

Here $7\frac{2}{3}-3\frac{3}{4}=7\frac{8}{12}-3\frac{9}{12}$; $\frac{9}{12}$ cannot be taken from $\frac{8}{12}$, we must therefore increase the $\frac{8}{12}$ by a unit, or $\frac{12}{12}$, borrowed from the 7, thus converting the operation into $6\frac{20}{12}-3\frac{9}{12}=3\frac{11}{12}$. And similarly in other such cases.

Exercises.

1. $3\frac{4}{5}-2\frac{3}{4}$.
2. $1\frac{6}{7}-\frac{5}{8}$.
3. $\frac{11}{12}-\frac{3}{7}$.
4. $\frac{13}{18}-\frac{5}{8}$.
5. $2\frac{4}{7}-1\frac{3}{5}$.
6. $4\frac{7}{10}-3\frac{10}{11}$.

7. $5\frac{13}{24}-4\frac{17}{18}$.
8. $4\frac{19}{20}-4\frac{15}{16}$.
9. $1\frac{17}{32}-\frac{19}{24}$.
10. $2\frac{13}{15}+3\frac{9}{11}-4\frac{5}{12}$.
11. $6\frac{7}{11}+1\frac{9}{33}-5\frac{8}{9}$.
12. $5\frac{7}{8}-6\frac{3}{4}+1\frac{5}{6}$.
13. $8\frac{5}{12}-3\frac{9}{16}-2\frac{7}{15}$.
14. $\frac{14}{21}+5\frac{4}{7}-6\frac{2}{15}$.
15. $10\frac{14}{15}-7\frac{8}{35}-2\frac{5}{12}$.
16. $4\frac{13}{55}-2\frac{7}{11}-1\frac{4}{5}$.
17. $3\frac{5}{6}-1\frac{4}{7}+5\frac{8}{9}-2$.
18. $9\frac{17}{18}-7\frac{2}{3}-1\frac{2}{5}+\frac{1}{2}$.
19. $11\frac{13}{16}-6\frac{7}{12}-2\frac{4}{9}-1\frac{1}{5}$.
20. $4\frac{17}{5}+\frac{9}{11}-\frac{7}{18}-3\frac{3}{10}$.
21. $5\frac{13}{8}-2\frac{7}{15}+3\frac{5}{18}+\frac{1}{2}$.
22. $7\frac{1}{4}-5\frac{1}{2}-\frac{7}{24}-\frac{15}{16}$.
23. $13\frac{12}{22}-12\frac{2}{11}-\frac{3}{8}-\frac{1}{12}$.
24. $15\frac{7}{9}+1\frac{1}{2}-14\frac{11}{12}+\frac{3}{5}$.

(57.) MULTIPLICATION OF FRACTIONS.

When anything is multiplied by a *fraction*, the operation performed is, in reality, something more than mere *multiplication*, in the sense in which that word is used when the multiplier is a whole number: thus, if anything is to be multiplied by $\frac{1}{5}$, the meaning is, that we are to take *one-fifth* of that thing *once;* if it is to be multiplied by $\frac{2}{5}$, the meaning is, that we are to take *one-fifth* of it *twice;* if by $\frac{3}{5}$, we are to take *one-fifth* of it *three times;* and so on. It is so far like ordinary multiplication, that it signifies *repetitions*, or the taking of a quantity a certain number of times: but then it is not the *whole* of this quantity, but only a part of it that is taken that number of times: the *part* to be taken is made known to us by the denominator of the multiplier; the *number of times* it is to be taken, by the numerator.

Suppose we have to multiply $\frac{3}{7}$ by $\frac{2}{5}$; the meaning is, that we have to take a *fifth part* of $\frac{3}{7}$ *twice:* now, by the *fifth part* of anything, is meant that thing divided by 5; in the proposed instance, it is $\frac{3}{7}$ divided by 5; but $\frac{3}{7}$, is 3 divided by 7, and if there be another division by 5, the result is of course the same as if 3 were at once divided by 5×7, or 35; so that a fifth part of $\frac{3}{7}$ is $\frac{3}{35}$; and, consequently, *two*-fifths must be $\frac{6}{35}$; we thus see, that $\frac{3}{7}\times\frac{2}{5}=\frac{6}{35}$, so that the product of the two fractions is nothing more than the product of their numerators divided by the product of their denominators; and, moreover, that $\frac{3}{7}$ *multiplied* by $\frac{2}{5}$, and $\frac{2}{5}$ths *of* $\frac{3}{7}$ths, or $\frac{3}{7}$ths *of* $\frac{2}{5}$ths, are one and the same in meaning. This must be kept in remembrance.

Although only a single instance is here taken for illustration, yet you will easily see, that the same reasoning would

apply to any two fractions whatever, and therefore that the following rule must be true for all cases.

RULE. Multiply the numerators together, and you will get the numerator of the product. Multiply the denominators together, and you will get the denominator of the product.

NOTE 1. If a multiplier be a *whole* number, it may be considered as a fraction with 1 for denominator: if a multiplier be a *mixed* number, it must be converted into an improper fraction.

2. It is very likely, that in a row of fractions to be multiplied together, there may be found *factors* in the numerators equal to factors in the denominators; if so, *cancel* them, or omit them in the multiplications; for there is no use in preserving common factors in numerator and denominator of the product, which ought always to be in the lowest terms. And, on this account, if a fraction is to be multiplied by a whole number, it is better to *divide* the denominator by the number, whenever the denominator exactly contains it.

Ex. 1. What is the product of $\frac{2}{3}$, $2\frac{1}{4}$, and $\frac{2}{7}$?

Reducing the mixed number to an improper fraction, we have $\dfrac{2}{3} \times \dfrac{9}{4} \times \dfrac{2}{7} = \dfrac{2 \times 3 \times 3 \times 2}{3 \times 2 \times 2 \times 7} = \frac{3}{7}$.

As here the factors, 2 and 2, furnished by the numerators, are the same as the factors of 4 in a denominator, these common factors are cancelled; also, since there is a factor 3 in the numerator 9, the same as a denominator, these two 3's are also cancelled; so that, after the cancellings, there remains only 3 in the numerator, and 7 in the denominator; and, therefore, the product is found without actually *multiplying* at all.

2. Multiply $\frac{4}{9}$ of $3\frac{1}{5}$ by $\frac{3}{7}$ of $2\frac{1}{2}$.

$$\frac{4}{9} \times \frac{16}{5} \times \frac{3}{7} \times \frac{5}{2} = \frac{2 \times 16}{3 \times 7} = \frac{32}{21} = 1\tfrac{11}{21}.$$

In working this example, I would recommend you to proceed thus: having written down all the given fractions, with the signs of multiplication between them, as above, and having put the sign of equality, draw the line that is to separate the resulting numerator and denominator, *before* putting anything above or below; then, looking at the first numerator, 4, examine the row of denominators; the last of this row, 2, you find to be a factor of 4, therefore expunge or cancel this factor, putting only the other factor 2 above

the line of separation, and draw your pen through the denominator 2, to remind you, when you come to it, that it is done with: then, looking at the denominator 9, of the first fraction, and glancing at the row of numerators, you see a 3, which being a factor of 9, you put only the other factor 3 below the line of separation, and draw your pen through the numerator 3, to show that it is cancelled. The next numerator, 16, has no factor belonging also to the uncancelled denominators; this 16, therefore, you put down as it is, against the number before put in the numerator's place; and as the denominator 5 of the second fraction is cancelled by the numerator 5 of the fourth, you draw your pen through both, and pass on to the next fraction, and you see that the only uncancelled number remaining is the denominator 7; you write this, therefore, in the denominators place, against the number already there, and put the multiplication sign between the two; and you thus have the numerator and denominator of the product free from useless factors; that is, you get the resulting fraction in its lowest terms.

3. Multiply $\frac{2}{3}$, $3\frac{1}{4}$, 5 and $\frac{3}{4}$ of $\frac{3}{5}$ together.

$$\frac{2}{3}\times\frac{13}{\underset{2}{4}}\times\frac{5}{1}\times\frac{3}{4}\times\frac{3}{5}=\frac{13\times3}{2\times4}=\frac{39}{8}=4\tfrac{7}{8}.$$

Here the first numerator 2 cancels a factor 2, in the next denominator 4, but as the other factor 2 remains uncancelled, you write the uncancelled 2 below the 4, which, with the numerator 2, is then crossed out: the first denominator 3 is also cancelled, with the fourth numerator, so that nothing as yet is put on the right of the sign =. Passing then to the numerator 13, you see that nothing below cancels it; you therefore put 13 in the numerators place, and the uncancelled 2 below it. Passing now to the third numerator, 5, you see that it is cancelled by the last denominator; there remains, therefore, only the uncancelled denominator of the fourth fraction, and the uncancelled numerator of the fifth.

In most books on Arithmetic, the cancelled figures, in examples of this kind, are printed with the cancelling marks across them; but this gives the work an unsightly appearance; and although I recommend this plan to you for your own private convenience, yet, in presenting your work to the inspection of another, I would not recommend it to be shown all defaced by these scratches: when they have served your purpose, they should be removed, or an undefaced copy of the work taken.

Exercises.

1. Multiply $\frac{4}{15}$ by $\frac{7}{24}$.
2. Multiply $3\frac{3}{7}$ by $2\frac{21}{22}$.
3. $\frac{3}{4}\times\frac{7}{9}\times\frac{2}{5}$ of $\frac{3}{8}$.
4. $\frac{1}{3}$ of $\frac{5}{7}$ of $\frac{6}{11}$.
5. $\frac{2}{9}\times4\frac{15}{16}\times\frac{2}{3}$ of 5.
6. $1\frac{3}{5}\times2\frac{4}{7}\times3\frac{1}{8}$.
7. $2\frac{11}{12}\times5\frac{9}{10}\times7$.
8. $66\frac{7}{9}\times\frac{13}{14}\times5\frac{3}{5}$.
9. $4\frac{3}{5}\times\frac{6}{7}$ of $3\frac{2}{3}$.
10. $\frac{2}{7}$ of $9\frac{3}{4}\times\frac{1}{5}$ of $3\frac{1}{2}$.
11. $\frac{2}{3}\times4\frac{1}{6}\times\frac{7}{9}$ of 5.
12. $\frac{5}{6}\times7\frac{3}{8}\times1\frac{2}{3}\times\frac{2}{7}$.
13. $3\frac{2}{7}$ of $4\frac{1}{5}\times\frac{7}{8}$ of 7.
14. $12\frac{8}{11}\times3\frac{2}{3}\times2\frac{1}{2}$ of $1\frac{3}{4}$.
15. $13\frac{1}{3}\times\frac{3}{10}\times1\frac{2}{7}\times\frac{8}{9}$.
16. $21\frac{2}{5}\times1\frac{1}{2}\times\frac{2}{9}\times\frac{2}{45}$.

(58.) DIVISION OF FRACTIONS.

You have already seen, that when you multiply by a proper fraction, the product is always *less* than the multiplicand; you will be prepared to expect, therefore, that when you divide by a proper fraction, the quotient will always be *greater* than the dividend. If your divisor is a *whole* unit, or 1, the quotient is only *equal* to the dividend; so that when your divisor is *less* than 1, that is, a proper fraction, the quotient must be *greater* than the dividend, for the less your divisor the greater the quotient, and if one divisor be a *third* part, a *fourth* part, &c., of another, the quotient due to that part must be *three* times, *four* times, &c., the quotient due to that other. All this is plain: suppose you have to divide by $\frac{3}{4}$, that is, the fourth part of 3, the quotient must be *four* times the quotient you would get by dividing by 3 itself; so that you would get the true quotient by first dividing by 3, and then multiplying what you would get, by 4. Division by a *fraction*, therefore, would thus imply both division and multiplication, for what is here said of the divisor $\frac{3}{4}$, you must see would equally apply to any other fractional divisor: the true quotient would always be got by dividing the dividend by the *numerator* of the divisor, and then multiplying the result by the *denominator*. For instance, if you had to divide $\frac{2}{3}$ by $\frac{5}{7}$, you would first divide the $\frac{2}{3}$ by 5; this would give you $\frac{2}{15}$; for 2, first divided by 3, and the result by 5, is, in fact, 2 divided by 15: but the *true* quotient must be 7 times the quotient $\frac{2}{15}$, because your *true* divisor, namely, $\frac{5}{7}$, is only a seventh part of the divisor 5; therefore, the true quotient is $\frac{2}{15}\times7$, or $\frac{14}{15}$:

you thus see, that the quotient of one fraction by another, is got by multiplying the numerator of the dividend by the denominator of the divisor, and the denominator of the dividend by the numerator of the divisor; or, which is the same thing, the quotient is got by turning the divisor upside down, and then proceeding as if the operation were multiplication instead of division. Hence the rule.

(59.) RULE. Invert the divisor, or make numerator and denominator change places, and then proceed as if it were multiplication. Mixed numbers are, of course, to be reduced to improper fractions, as before.

NOTE. Before inverting the divisor, see whether the numerators of both fractions have a common factor; if they have, expunge it: see if the denominators have a common factor; if they have, expunge it. Remember to do this before applying the rule.

Ex. 1. Divide $\frac{5}{18}$ by $\frac{7}{12}$.

Here $\frac{5}{18} \div \frac{7}{12} = \frac{5}{3} \div \frac{7}{2} = \frac{5}{3} \times \frac{2}{7} = \frac{10}{21}$, the factor 6, common to both denominators, being cancelled before applying the rule, because if allowed to remain, it would enter numerator and denominator of the result.

2. Divide $5\frac{8}{15}$ by $2\frac{1}{6}$.

Here $\frac{83}{15} \div \frac{13}{6} = \frac{83}{5} \div \frac{13}{2} = \frac{83}{5} \times \frac{2}{13} = \frac{166}{65} = 2\frac{36}{65}$.

3. Divide $\frac{7}{22}$ of $8\frac{1}{4}$ by $\frac{3}{5}$ of $\frac{5}{9}$.

First, $\frac{7}{22} \times \frac{33}{4} = \frac{21}{8}$. Second, $\frac{3}{5} \times \frac{5}{9} = \frac{1}{3}$.

Then, $\frac{21}{8} \div \frac{1}{3} = \frac{21}{8} \times 3 = \frac{63}{8} = 7\frac{7}{8}$.

In inverting the divisor here, the denominator 1 is omitted as useless. If the divisor had been 3 instead of $\frac{1}{3}$, the divisor inverted would have been $\frac{1}{3}$, since 3 is $\frac{3}{1}$; and $\frac{1}{3}$ would have been called the *reciprocal* of 3; the *reciprocal* of any number being 1 divided by that number: the reciprocal of a fraction is merely that fraction inverted; thus, $1 \div \frac{2}{5} = 1 \times \frac{5}{2} = \frac{5}{2}$; $1 \div \frac{3}{7} = 1 \times \frac{7}{3} = \frac{7}{3}$, and so on; so that the rule for division of fractions may be stated thus: multiply the dividend by the *reciprocal* of the divisor, and you will get the quotient.

Exercises.

1. $\frac{3}{7} \div \frac{11}{28}$.
2. $\frac{14}{15} \div \frac{16}{27}$.
3. $2\frac{13}{14} \div 4\frac{7}{8}$.
4. $3\frac{13}{35} \div 7\frac{2}{15}$.
5. $1\frac{7}{10} \div 10\frac{1}{5}$.
6. $27 \div \frac{8}{15}$.
7. $\frac{3}{5}$ of $\frac{2}{7} \div \frac{5}{8}$.
8. $\frac{2}{3}$ of $1\frac{4}{9} \div 3\frac{7}{11}$.
9. $\frac{3}{7}$ of $2\frac{1}{2} \div \frac{1}{2}$ of $2\frac{3}{4}$.
10. $4\frac{5}{9} \div \frac{3}{8}$ of $\frac{4}{5}$ of 6.
11. $18 \div \frac{6}{11}$ of $\frac{2}{3}$ of $\frac{1}{4}$.
12. $\dfrac{3}{7\frac{3}{8}} \div \dfrac{2\frac{1}{4}}{3\frac{3}{8}}$.

NOTE. A fraction like either of the fractions in Ex. 12 is

called a *complex* fraction; the word *complex* implying that *one* of the terms of the fraction, at least, does *itself* contain a fraction. Each of the preceding exercises might have been written as a complex fraction, so as to dispense with the sign $\div$; and this way of indicating division is often the more convenient, as will be seen in the next article: before proceeding to which, however, it may be well to state the two following obvious inferences from the rules for multiplication and division in a distinct form, on account of the frequent application of them.

1. When a fraction is to be multiplied by a whole number, the correct product is obtained, whether we multiply the numerator by the number and leave the denominator untouched, or divide the denominator and leave the numerator untouched; the latter way, when the denominator is divisible by the number, is generally to be preferred. If the multiplier be *equal* to the denominator, the product is simply the numerator.

2. When a fraction is to be divided by a whole number, the correct quotient is obtained whether we divide the numerator by the number, leaving the denominator untouched, or multiply the denominator and leave the numerator untouched; the former when practicable is generally to be preferred.

(60.) *To find what Fraction one Quantity is of another Quantity.*

The fraction that one quantity is of another is expressed by writing the former as numerator, and the latter as denominator; both quantities, if *concrete*, being first reduced to the same denomination.

Ex. 1. What fraction of $\frac{2}{3}$ is $\frac{1}{4}$?

Here we have only to divide $\frac{1}{4}$ by $\frac{2}{3}$; therefore $\frac{1}{4}\times\frac{3}{2}=\frac{3}{8}$; so that $\frac{1}{4}$ is $\frac{3}{8}$ of $\frac{2}{3}$.

2. What fraction of $12\frac{3}{5}$ is 7?

$$\frac{7}{12\frac{3}{5}}=\frac{35}{63}, \text{ the fraction required.}$$

3. What fraction of £1 is 7*s.* 8*d.*?

Reducing these to the same denomination, we have $\frac{7\frac{8}{12}s.}{20s.}=\frac{92}{240}=\frac{23}{60}$; that is, 7*s.* 8*d.* is the $\frac{23}{60}$ths of £1. If $\frac{8}{12}$ had been replaced by $\frac{2}{3}$, the result would have come out in the lowest terms at once.

4. What fraction of $8s.\ 4d.$ is $3s.\ 9d.$?

As $4d.=\frac{1}{3}s.$, and $9d.=\frac{3}{4}s.$, we have $3\frac{3}{4}\div 8\frac{1}{3}=\frac{15}{4}\div\frac{25}{3}=\frac{3}{4}\div\frac{5}{3}=\frac{9}{20}$; so that $3s.\ 9d.$ is $\frac{9}{20}$ths of $8s.\ 4d.$

Exercises.

1. What fraction of $5s.$ is $1s.\ 10d.$?
2. What fraction of $2s.\ 6d.$ is $8\frac{1}{2}d.$?
3. Reduce $3s.\ 11d.$ to the fraction of a guinea.
4. Reduce $\frac{3}{7}$ of $2s.$ to the fraction of £1.
5. What fraction of a yard is 2 ft. 5 in.?
6. What fraction of a ton is 3 cwt. 1 qr.?
7. Reduce $\frac{9}{11}$ of 10 minutes to the fraction of an hour.
8. What fraction of $2\frac{3}{5}$ is $\frac{1}{3}$ of $3\frac{2}{7}$?
9. What fraction of £$1\frac{3}{7}$ is $7\frac{2}{5}s.$?
10. What fraction of $90°$ is $12°\ 23'\frac{1}{2}$?
11. What fraction of a week is 2 d. 17 h.?
12. What fraction of $1\frac{1}{2}$ is $\frac{7}{15}$ of $2\frac{1}{2}$?
13. What fraction of $2\frac{1}{2}$ is $\frac{7}{15}$ of $1\frac{3}{4}$?
14. What fraction of £7 $13s.\ 4\frac{1}{2}d.$ is £2 $14s.\ 2\frac{1}{4}d.$?
15. Reduce 3 roo. 21 po. 3 yds. to the fraction of 11 ac. 2 roo. 6 po.

(61.) *Examples of Multiplication and Division of concrete Quantities.*

Though the rules given for multiplying and dividing by a fraction or mixed number apply generally, whatever the things multiplied or divided may be, yet as these rules have been actually applied hitherto only to abstract numbers, it may be proper to give a few examples in which concrete quantities are concerned.

Ex. 1. Multiply $7s.\ 1\frac{1}{2}d.$ by $\frac{115}{9}$, or $12\frac{7}{9}$.

$$\begin{array}{rrl} & s. & d. \\ & 7 & 1\frac{1}{2}\times 12 \\ & & 12 \\ \hline 4 & 5 & 6 \\ 0 & 5 & 6\frac{1}{2} \text{ for } \frac{7}{9} \\ \hline £4 & 11 & 0\frac{1}{2} \\ \hline \end{array} \qquad\qquad \begin{array}{rl} s. & d. \\ 7 & 1\frac{1}{2}\times\frac{7}{9} \\ & 7 \\ \hline 9)49 & 10\frac{1}{2} \\ \hline 5 & 6\frac{1}{2} \\ \hline \end{array}$$

Or, since $7s.\ 1\frac{1}{2}d.=7\frac{1}{8}s.=\frac{57}{8}s.$, we have $\frac{57}{8}s.\times\frac{115}{9}=\frac{19}{8}s.\times\frac{115}{3}=\frac{2185}{24}s.=91\frac{1}{24}s.=£4\ 11s.\ 0\frac{1}{2}d.$ The best way to multiply by 19, is to multiply by 20 and subtract the multiplicand.

2. Multiply 3 ton 7 cwt. 12 lb. by $2\frac{3}{4}$.

	ton.	*cwt.*	*qr.*	*lb.*
4)	3	7	0	12
				2
	6	14	0	24
for $\frac{1}{4}$	0	16	3	3
for $\frac{1}{2}$	1	13	2	6
	9	4	2	5

The product by $\frac{3}{4}$ may also be got by taking $\frac{1}{4}$ from the whole, thus:

	ton.	*cwt.*	*qr.*	*lb.*	
4)	3	7	0	12	
	0	16	3	3	
	2	10	1	9	for $\frac{3}{4}$

3. Divide £7 11*s.* $6\frac{1}{2}$*d.* by $5\frac{2}{13}$.

This is to multiply by $\frac{13}{67}$, which may be done by first reducing to the *lowest* denomination, but more easily as below.

£. *s.* *d.*
7 11 $6\frac{1}{2}$
20
151
12
1818
4
7274
13
21822
7274
67) 94562 (1411$\frac{25}{67}$ farthings.
67
275
268
76
67
92
67
25

4) 1411 + $\frac{25}{67}$
12) $352\frac{3}{4}$
2,0) 2,9 4
£1 9 $4\frac{3}{4}+\frac{25}{67}f.$

Or thus: £. *s.* *d.*
7 11 $6\frac{1}{2}$, to be *added.*
12+1=13
90 18 6
67) 98 10 $0\frac{1}{2}$ (£1 9*s.* $4\frac{3}{4}$*d.* $+\frac{25}{67}f.$
67
31
20
630
603
27
12
324
268
56
4
226
201
25

Exercises.

1. Multiply £12 11*s*. 6*d*. by $12\frac{3}{4}$.
2. What is the value of $\frac{3}{7}$ acre at £$2\frac{3}{11}$ per acre?
3. Multiply 17*s*. $5\frac{1}{4}$*d*. by $754\frac{3}{4}$.
4. A person walks $77\frac{2}{5}$ miles in $10\frac{1}{2}$ hours, at what rate is that per hour?
5. Seven pieces of cloth, each containing $11\frac{3}{8}$ yards, cost £$54\frac{3}{8}$, what was the price per yard?
6. Three persons, A, B, C, purchase property worth £625; A purchases $\frac{3}{7}$, B $\frac{2}{9}$, and C the rest: what is the worth of each person's share?
7. The total amount of gold exported from San Francisco, in California, during the month of October, 1851, was 6884875 American dollars; what was its worth in English money, an American dollar being $\frac{9}{10}$ of an English one? [American dollars differ in value in different States.]
8. A person has $\frac{5}{8}$ of a cargo, worth £900, and wishes to sell $\frac{2}{3}$ of his share; what is it worth?
9. A person holding $\frac{5}{9}$ of property worth £864 10*s*. wishes to dispose of as much of it as will produce £160; what share will he possess after the sale?
10. The true length of the year is very nearly 365 days 5 hours 49 minutes; what is the length of the $\frac{7}{52}$ part, or 7 weeks, considering a week to be the 52nd part of a year?

(62.) *To find the Greatest Common Measure of two or more Whole Numbers.*

By the greatest common measure of a set of numbers is meant the greatest number which will divide them all without leaving any remainder; a number being said to *measure* another only when it is contained in that other a certain number of times exactly: 3, for instance, measures 9, 12, 15, &c., but it does not measure 10, 13, 16, &c. Of the numbers 8 and 12, 4 is the *greatest* common measure; that is, the greatest divisor common to them both: 6 is the greatest common measure of 12 and 18, as also of 12, 18, and 36; and so on. The words *greatest common measure* are, for shortness, replaced by the letters G. C. M.

The chief use of a rule for finding the G. C. M. of two numbers is to enable us to discover with certainty the *lowest terms*

in which any proposed fraction can be written. In the fractions hitherto considered, this object has been attained by a simple inspection of the numerator and denominator, as the factors common to both are often sufficiently obvious to become readily detected by a little examination; in which examination the table of factors at the end will frequently be of assistance. I have postponed the general rule about to be given till now, because you should always endeavour to discover the greatest common measure in this way, without resorting to any such rule; for the operation by it is sometimes long, though not difficult, and frequently ends by showing that no common measure exists. Moreover, it is not indispensably necessary that a fraction be reduced to its lowest terms: factors which *obviously* enter both numerator and denominator should always be removed; but when the detection of them requires the aid of the rule below, they might, in most cases, be suffered to remain, without bringing any discredit upon the computer; for the simplified result is seldom worth the trouble of obtaining it. It is right, however, that you should know the rule, more especially as it may be applied to other purposes.

(63.) RULE. To find the G. C. M. of two numbers, divide the greater by the less; make the remainder a new divisor, and the former divisor a new dividend; then make the second remainder a divisor, and the preceding divisor a dividend, and so on, always dividing the last divisor by the last remainder till the remainder disappears, or becomes 0. The divisor which thus leaves no remainder is the G. C. M. of the two numbers. If there be a *third* number, apply the rule to the G. C. M. of *two* and the third : and so on.

NOTE.—You may very often stop the process before it terminates in this way of itself. The divisors continually diminish in magnitude, and on this account, though the original numbers may be too large for you to detect their G. C. M. without a rule, yet the smaller numbers, which form the several divisors, may be small enough, after a step or two, to enable you to see by simple inspection what the G. C. M. of a pair of *them* is, and it so happens, that the G. C. M. of any two of the divisors is always likewise the G. C. M. of the original numbers. And it is strange that this way of curtailing the work is not expressly pointed out in books of arithmetic. In the first example on next page, the process is carried on till it terminates of itself, for the purpose of showing you all the steps in full; but after the first of these steps, the remainder of the work is unnecessary. Your first remainder 92 will obviously divide by 2; the quotient 46 will also divide by 2, giving 23; the only different factors of 92 are therefore 2 and 23. Now 161 will not divide by 2; but you find, upon trial, that it will divide by 23. Hence 23 is the G. C. M.

Ex. 1. Find the G. C. M. of 161 and 253; and thence reduce the fraction $\frac{161}{253}$ to its lowest terms.

Proceeding by the rule, as in the margin, the G. C. M. is found to be 23; we infer, therefore, that both numerator and denominator of the fraction have 23 for their highest common divisor. Dividing them, therefore, by this number, we find the fraction in its lowest terms to be $\frac{7}{11}$, which is certainly a much simpler form than $\frac{161}{253}$. And I may here observe, that we can often form a much more correct estimate of the value of a fraction after it is reduced to its lowest terms: you can form a better notion of 7 elevenths of a thing than of 161 parts of it out of 253.

```
161)253(1
    161
    ---
     92)161(1
         92
         --
         69)92(1
            69
            --
            23)69(3
               69
               --
```

2. Find the G. C. M. of 175 and 912.

Here we find that the second divisor is 37, which obviously has no factor, unity not being considered as a factor; hence the numbers have no common measure, so that the fraction $\frac{175}{912}$ is already in its lowest terms.

```
175)912(5
    875
    ---
     37
```

3. Reduce $\frac{12321}{54345}$ to its lowest terms.

Proceeding as in the margin, we arrive, after a few steps, at the remainder 210, a number whose factors, it is very easy to see, are 10, 7, and 3, the 10 being composed of the factors 2 and 5; so that the G. C. M., if any common measure exist, must be one or more of these factors. A glance at the last figure of 663 shows that neither 2 nor 5 can be a divisor of it. It is easy to try 7, which is found not to succeed; but 3 does succeed: hence the G. C. M. of the terms of the proposed fraction is 3. And thus several steps of work are saved. Dividing numerator and deno-

```
12321)54345(4
      49284
      -----
       5061)12321(2
            10122
            -----
             2199)5061(2
                  4398
                  ----
                   663)2199(3
                       1989
                       ----
                        210
```

(*See Key*, *p.* 61.)

minator by 3, we have $\frac{12321}{54345}=\frac{4107}{18115}$, which, after all, is a degree of simplification too trifling to be worth the trouble by which it has been effected; but then this could not have been foreseen. The common measure 3 could have been easily found by inspection.

(64.) The truth of the rule for the G. C. M. depends upon two general principles; namely, 1. *If one number be divided by another, every factor common to dividend and divisor, must also be a factor of the remainder.* For suppose two numbers, that have *no* factor in common, to be divided one by the other, and the remainder to be found: it is plain, that if the dividend and divisor were each to be multiplied by any number, 4, for instance, and the division *then* to be performed, we should get the same *quotient* as before; but the *remainder* would be 4 times the former remainder, because if you subtract 4 times a number from 4 times another number, the remainder must be 4 times as great as it would have been if you had subtracted without multiplying the numbers by 4. You see, therefore, that when a factor is introduced into dividend and divisor, it is also introduced into the remainder; so that whenever a common factor exists in dividend and divisor, you would be sure to find it out by trying *all* the factors of the remainder.

2. *Whatever factor is common to remainder and divisor, also belongs to the dividend.* For the dividend is *equal* to the product of quotient and divisor, with the remainder added: the portion of the dividend, furnished by the product of quotient and divisor, has, of course, whatever factor the divisor has; so that if the other portion, namely, the remainder, have *the same* factor, it follows, that the sum of these portions, that is, the dividend itself, must have that factor. These two principles suggest the rule; thus, referring to Example 1, the greatest common factor of the remainder 92 and divisor 161, is also the greatest common factor of 161 and 253: in like manner, the G. C. M. of 69 and 92, is the G. C. M. of 92 and 161, and, *consequently*, from what has just been inferred, the G. C. M. of 161 and 253: and, lastly, the G. C. M. of 23 and 69, which we see is 23 itself, is the G. C. M. of 69 and 92, and, therefore, of 92 and 161, and of 161 and 253.

Exercises.

1. Find the G. C. M. of 247 and 323.
2. Of 272 and 425.
3. Of 57 and 63.

4. Of 408 and 527.
5. Of 1164 and 1261.
6. Reduce $\frac{297}{1021}$ to its lowest terms.
7. Reduce $\frac{618}{816}$ to its lowest terms.
8. Reduce $\frac{7163}{15863}$ to its lowest terms.
9. Find the G. C. M. of 5283 and 176491.
10. A field is 169 rods long, and 156 rods wide: what is the length of the longest chain that will exactly measure both length and width?
11. Three persons, A, B, C, having, respectively, £323, £456, and £551, lay it out in land, at the greatest price per acre that will allow each to spend the whole of his money: what was the price per acre, and how many acres did each buy?
12. Find the least number of ounces of standard gold that can be coined into an exact number of half-sovereigns: standard gold being £3 17*s.* 10½*d.* an ounce.*

(65.) *To find the Least Common Multiple of a Set of Numbers.*

Any number that is exactly divisible by a set of other numbers is called *a common multiple* of those others, and the *least* number that is exactly divisible by a set of others, is called the *least* common multiple of those others: it is, for brevity, expressed by the initial letters, L. C. M.

From knowing how to find the L. C. M. of a row of numbers, we can always reduce a row of fractions to others equal to them in value, and, at the same time, having the lowest possible common denominator; for the lowest common denominator will, of course, be the least common multiple of the original denominators. Each changed fraction will have this L. C. M. for denominator, and therefore for numerator, it must have the product that arises from dividing the L. C. M. by the original denominator, and multiplying the original numerator

* As a hint towards the management of this question, the learner may be informed, that if the £3 17*s.* 10½*d.* be reduced to half-pence, and the number of half-pence be divided by the number in 10*s.*, the quotient will give the number of half-sovereigns and the fraction of a half-sovereign that can be coined from one ounce; he will then have to find the lowest number the dividend must be multiplied by to prevent the entrance of a fraction in the quotient. It will not be difficult for him to see that £3 10*s.* out of the £3 17*s.* 10½*d.* may be neglected in the work. He will find the least number of ounces to be 80, and the corresponding number of half-sovereigns to be 623.

by the quotient, because it is only by such multiplication that the original numerator and denominator come to be *both* multiplied by the same number.

Before giving you a general rule for finding the L. C. M., it is, perhaps, better that I should show you, by an example, what appears to me to be the most convenient and obvious method of proceeding. Suppose the numbers 24, 10, 9, 32, 6, 45, and 25, were proposed, we should have to discover the lowest number that would give a quotient free from fractions, when divided by any one of these seven numbers. Let us first divide the 24 by the 10, we shall have $\frac{24}{10} = 2\frac{2}{5}$; and as the fraction $\frac{2}{5}$ is in its lowest terms, with 5 for denominator, it is plain that we must take the dividend 5 times, *at least*, in order to render the quotient free from a fraction; so that 24×5, or 120, is the least number which is exactly divisible by *both* 24 and 10. Again; $\frac{120}{9} = 13\frac{1}{3}$, so that we must multiply the 120 by 3, *at least*, to make the quotient by 9 a whole number: hence, 360 is the least number exactly divisible by 24, 10, and 9. Again; $\frac{360}{32} = 11\frac{1}{4}$, so that $360 \times 4 = 1440$, is the least number divisible by 24, 10, 9, and 32: we need not attend to the 6, because whatever is divisible by 24 is also divisible by 6:* taking, therefore, the next number, we have $\frac{1440}{45} = 32$, so that 1440 is the least number divisible by 24, 10, 9, 32, 6, and 45. Lastly, $\frac{1440}{25} = 53\frac{3}{5}$; consequently, $1440 \times 5 = 7200$, is the L. C. M. of all the numbers.

From this example you will be prepared for the rule I propose to give: it is as follows:—

RULE. Take any two of the numbers for dividend and divisor. If the quotient have a fraction, reduce it to its lowest terms, and multiply the dividend by its denominator. Take the product for a new dividend, and another of the numbers for divisor: if the quotient have a fraction, reduce it to its lowest terms, and, as before, multiply the dividend by the denominator: take the product for a new dividend, and another of the numbers for a divisor; and so on, till all the proposed numbers have been used, omitting those which are obviously contained in any of the others, or in a dividend already found: the last product will be the L. C. M.

* As 45 is one of the given numbers, we might, for a like reason, have neglected the 9; since whatever is divisible by 45 must be divisible by 9: or we might have neglected the 45, since whatever is divisible by 10, and 9, must obviously be divisible by 45.

NOTE 1. If any quotient occur without a fraction, the same dividend is to be used with the *next* divisor.

2. When in going over the row of numbers you come to one which is a *prime* number (page 20), you should see, as you may readily do, whether this prime number be contained exactly in any of the preceding: if it be, you are to omit it, as the rule directs; but if it be not, then you will know that the quotient, arising from dividing your last dividend by it, must have a fraction with that prime number for denominator: you need not, therefore, be at the trouble of performing the division; you will merely have to multiply the dividend by the prime number, and then to pass on to the next number; but a still better way will be, to reserve these prime numbers, after having selected them from the entire row, and to apply the rule only to the composite numbers into which the primes do not enter; and when you have got the L. C. M. of the composite numbers, to multiply it by the primes, one after another; of course, you are to take no account at all of such primes as are contained in any of the composite numbers.

Exercises.

Find the least common multiple of

1. 8, 12, 18, 20	4. 24, 16, 20, 30, 25
2. 3, 9, 27, 81	5. 27, 24, 15, 126
3. 2, 3, 4, 5, 6, 7, 8, 9	6. 242, 748, 21, 427.

It may be as well to remind you here, that as 6 and 8, in Ex. 3, contain 2, 3, and 4, these latter numbers may be *omitted;* and as 5 and 7 are *primes*, not contained in any of the other numbers, they may be *reserved* (NOTE 2); so that you need apply the rule only to 6, 8, 9, of which the L. C. M. is 72: therefore, $72 \times 5 \times 7 = 2520$, the L. C. M. of the proposed numbers. The following example is also one in which you may, with advantage, avail yourself of NOTE 2.

7. Find the L. C. M. of 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21.

Here there are four prime numbers, which you may reserve till you have applied the rule to the others.

From what has now been said about the L. C. M. of a set of numbers, you can never be at a loss how to reduce a row of fractions to others equal to them, and having the least possible common denominator: as observed at the commencement of Art. 65, when you have found the L. C. M. of the deno-

minators, by aid of the above directions, you will only have to divide it by the denominator of the first fraction, to get the number by which the numerator must be multiplied, in order that you may have the proper numerator of the corresponding changed fraction; and so of all the following fractions. In other books on Arithmetic, you will see rules for finding the L. C. M. different from that above: some of these may *look* shorter, but I think the one here given will be found as expeditious as any, while it is much more easy of proof, and of being borne in the memory. (*See Key, p.* 61.)

PRACTICE.

(66.) PRACTICE is the name given to a set of operations in commercial arithmetic, by which the prices of commodities may often be calculated in an expeditious manner.

From the examples of these operations which I shall here give, you will easily see what the peculiar method called Practice really is; and you will, at the same time, observe, that though it does not enable you to do anything you cannot do already by means of the rules for compound quantities before taught, yet, that in many cases, both time and figures are saved by replacing these rules by PRACTICE; and such a saving is matter of consideration in actual business. When you are familiar with the method, it will be for you to determine, in any example you may have to work, whether the former methods, or the method of Practice, or a combination of these two, be the most likely to save time and trouble.

Ex. 1. What is the value of 48 tons, at £6 10*s*. a ton?

Here, instead of reducing the money to shillings, and then multiplying by 48, we should regard the £6 10*s*. as $6\frac{1}{2}$, working the example by multiplying the 48 by 6, and adding the half of 48 to the product: the sum is, of course, the whole product of 48 by $6\frac{1}{2}$, and the result comes out at once in pounds, without any reduction, as in the margin. The operation so conducted and so arranged is an operation of PRACTICE.

		48
		6
		288
10*s*.	$\frac{1}{2}$	24
		£312

2. Suppose the price per ton had been £6 15*s.*: then, observing that 10*s.* is the half of a pound, and 5*s.* the half of 10*s.*, the operation would have been as here annexed. Here you see, that 48 times the £6 is £288; 48 times the $\frac{1}{2}$£, or 10*s.*, is £24; and as 48 times 5*s.* must be the half of this, we get the additional £12: so that the whole sum is £324.

		48
		6
		288
10*s.*	$\frac{1}{2}$	24
5*s.*	$\frac{1}{2}$	12
		£324

You thus see, that by working examples of this kind by Practice, you avoid the trouble of reducing the compound quantity concerned to the lowest denomination, which appears in it, and then of reducing the result back again. In Practice the highest denomination is preserved, and the lower denominations considered as Aliquot Parts, that is, exact measures of the higher; so that the chief thing you have to do, is to consider how these lower denominations may be most conveniently cut up into aliquot parts of the higher: thus, in the last example, the 15*s.* was cut up into 10*s.*, the half of the highest unit, a £, and into 5*s.*, the half of the 10*s.*, or the fourth of the £.

3. Suppose the price had been £6 17*s.* 6*d.* per ton: then the aliquot parts would have been $\frac{1}{2}$ of a £ for 10*s.*, $\frac{1}{2}$ of 10*s.* for 5*s.*, and $\frac{1}{2}$ of 5*s.* for the remaining 2*s.* 6*d.*: the work would therefore have stood as in the margin. The fractions are put against the several sums into which the shillings and pence are cut up, to show what aliquot parts they are; but you must avoid the error of saying you *divide* by $\frac{1}{2}$, $\frac{1}{4}$, &c., for you divide by 2, 4, &c., in order to get the half, the fourth part, &c. It is to prevent your falling into this error, that each fraction is placed, not beside the *dividend*, but beside the *quotient*.

		48
		6
		288
10*s.*	$\frac{1}{2}$	24
5*s.*	$\frac{1}{2}$	12
2*s.* 6*d.*	$\frac{1}{2}$	6
		£330

4. Suppose, lastly, the price were £6 7*s.* 10$\frac{1}{2}$*d.* Here you may cut up the shillings and pence as follows: 5*s.* is $\frac{1}{4}$£; 2*s.* 6*d.* is $\frac{1}{2}$ of 5*s.*; 3*d.* is $\frac{1}{10}$ of 2*s.* 6*d.*, or 30*d.*; and, lastly, 1$\frac{1}{2}$*d.* is $\frac{1}{2}$ of 3*d.* Hence the work is as in the margin.

		48	
		6	
		288	
5*s.*	$\frac{1}{4}$	12	
2*s.* 6*d.*	$\frac{1}{2}$	6	
3*d.*	$\frac{1}{10}$	0	12
1$\frac{1}{2}$*d.*	$\frac{1}{2}$	0	6
		£306	18

5. If the price had been £6 8*s.* 10½*d.*, the work might have been conducted as here annexed, from which you will see that there is sometimes need for a little reflection as to the most convenient set of aliquot parts.

When you come to the bottom of this page you will see how, by a little contrivance, the work here given in the margin may be abridged. I give you the operation in this form at first, as also that of the next example, in order that you may become acquainted with the management of aliquot parts.

		48	
		6	
		288	
6*s.* 8*d.*	$\frac{1}{3}$	16	
1*s.* 4*d.*	$\frac{1}{5}$	3	4
8*d.*	$\frac{1}{2}$	1	12
2*d.*	$\frac{1}{4}$		8
½*d.*	$\frac{1}{4}$		2
		£309	6

6. Find the value of 76 at 13*s.* 11¾*d.* each.

In this example, the highest denomination being shillings, the first aliquot part taken is that of a shilling; for the portion of the price equal to one shilling, the 76 articles would be 76*s.*; therefore, for half that portion, that is, for the 6*d.*, the sum is 38*s.* The total amount of all the component portions of the entire value is 1062*s.* 5*d.*: therefore, dividing the shillings by 20, we have £53 2*s.* 5*d.**

		76	
		13	
		228	
		76	
		988	
6*d.*	$\frac{1}{2}$	38	
3*d.*	$\frac{1}{2}$	19	
2*d.* of 6*d.*	$\frac{1}{3}$	12	8
¾*d.* of 3*d.*	$\frac{1}{4}$	4	9
		1062	5

Sometimes you may abridge the work by increasing the price of the single article, and then correcting the result, by subtracting what is due to the increase: for instance, when the price is 17*s.* 6*d.*, preceded by pounds or not, you may increase it by 2*s.* 6*d.*, and then allow for the overplus, by subtracting the

		48 at £6 10*s.*
		6
		288
10*s.*	$\frac{1}{2}$	24
		£312

* As noticed above, the operation in the margin merely shows what the steps suggested by the rule of Practice really are; but the more completely the *principles* of arithmetic are taught, the more independent of *rules* does the learner become. He who knows these principles well, need never ransack his memory for *rules.* The value of the 76 articles above, at 14*s.* each is 1064*s.*; this diminished by 76 farthings, or 19 pence, that is 1*s.* 7*d.*, leaves 1062*s.* 5*d.*, as above.

amount for $\frac{1}{8}$£: Ex. 3 might have been worked in this way: but I shall illustrate what I mean by taking Ex. 5. Here we see, that by increasing the price by 1*s.* 1½*d.*, it becomes £6 10*s.*: the correct result will therefore be found by computing the value of 48 articles at £6 10*s.*, as at the bottom of last page, and subtracting from it the value of 48 at 1*s.* 1½*d.*

		48 at 1*s.* 1½*d.*
1½*d.*	$\frac{1}{8}$	6
		54*s.*
		£312—£2 14*s.* =
		£309 6*s.*

I shall work out for you only one more example, serving to show how you may sometimes combine Practice with Compound Multiplication to advantage: you will see that one part of the operation is performed by both methods.

7. Find the value of 7 cwt. 3 qrs. 11 lbs., at £2 13*s.* 1*d.* per qr.

As the price is per qr., we reduce the 7 cwt. 3 qrs. to 31 qrs., so that we shall have to multiply the price by 31, and then to take parts for the odd 11 lbs.: the first of these operations is here given in two distinct forms.

£.	*s.*	*d.*	
2	13	1×1	
		10	
26	10	10	
		3	
79	12	6	
2	13	1	
£82	5	7	...for 31 qrs...

		£.	*s.*	*d.*	
		31 at 2	13	1	
		2			
		62			
10*s.*	$\frac{1}{2}$	15	10		
2*s* 6*d.*	$\frac{1}{4}$	3	17	6	
6*d.*	$\frac{1}{5}$		15	6	
1*d.*	$\frac{1}{6}$		2	7	
		£82	5	7	
		1	0	10¼	for 11 lbs.
		£83	6	5¼	the value sought.

		£.	*s.*	*d.*
		2	13	1
7 lbs.	$\frac{1}{4}$		13	3¼
4 lbs.	$\frac{1}{7}$		7	7
		£1	0	10¼

Of the two ways of finding the £82 5*s.* 7*d.*, the first is to be preferred as the easier.

NOTE 1. It is worth while to observe, that whenever the price of a number of articles at an *even* number of shillings each is to be computed, the shortest way is to take only *half* the number of shillings, to multiply the given number by this half, and put down the *double* of the first figure of the product for *shillings*, the remaining part of the product will be

pounds: thus, if you have to compute the value of 123 articles at 18*s.*, the work is comprised in only the few figures in the margin. By doubling the first figure 7 of the product, for shillings, and not doubling the other figures, you get the same result as you would do if you were to multiply by 9 and by 2, and were then to divide the resulting number of shillings by 20, as is sufficiently obvious.

123 at 18*s.*	
9	
£110	14*s.*

NOTE 2. It may also be noticed here, that in the purchase of some kinds of goods, certain *commercial allowances* are made for the packages, chests, &c. containing them, as also for waste; and these allowances are deducted from the *gross weight*. The deduction for the package is called *tare*, and is generally at so much per cwt. This deduction is made first. The deduction for waste, which is made next, is called *tret*, and is an allowance of $\frac{1}{26}$, or $\frac{1}{2}$ of $\frac{1}{13}$ of the weight, when diminished by the tare. Besides these, a trifling allowance is sometimes made to retailers, for what is called "the turn of the scale;" it goes by the name of *cloff*. The amount of these deductions may always be computed by Practice; and no special directions are necessary for executing the work: the deductions being made, the result is the *net weight*.

*Exercises.**

			£.	*s.*	*d.*				£.	*s.*	*d.*
1.	37	at	2	16	6	7.	243	at	2	4	$7\frac{1}{4}$
2.	41	at	3	17	8	8.	317	at	4	3	$9\frac{1}{2}$
3.	79	at	5	11	7	9.	353	at	7	18	4
4.	83	at	6	3	$6\frac{1}{2}$	10.	417	at	3	9	2
5.	133	at	1	13	$8\frac{1}{2}$	11.	358	at	6	11	5
6.	211	at	3	9	$5\frac{3}{4}$	12.	519	at	7	19	10

13. 4 cwt. 2 qr. 5 lb. at 16*s.* 4*d.* per cwt.
14. 2 cwt. 1 qr. 13 lb. at £1 13*s.* 9*d.* per cwt.
15. 38 yds. 1 ft. 7 in. at £2 3*s.* $7\frac{1}{2}$*d.* per yard.
16. 17 yds. 2 ft. 8 in. at 13*s.* $3\frac{1}{2}$*d.* per foot.
17. 7 oz. 11 dwt. 18 gr. at 3*s.* $8\frac{1}{4}$*d.* per oz.
18. 14 weeks $3\frac{1}{2}$ days at £1 12*s.* 6*d.* per week of 7 days.
19. 3 months 3 weeks 3 days at £1 3*s.* 10*d.* per week of 6 days.
20. 127 ac. 3 roo. 37 per. at £3 6*s.* 8*d.* per acre.
21. 17 cwt. 3 qr. at £5 5*s.* per ton.
22. 86 lb. 3 oz. 15 dwt. 18 gr. (troy) at £4 16*s.* 4*d.* per oz.
23. The *Sydney Morning Herald* (Aug. 1851) reports, that gold to the amount of from £5000 to £10000 reaches Sydney, in Australia, daily, from the gold fields:

* In working these exercises, the results are to be brought out to the nearest farthing, and are not to be encumbered with fractions of a farthing.

£3 8*s*. 4*d*. per ounce has been offered for all the gold Government might receive during two months: among the recent arrivals from Ophir was a lump weighing 51 oz. 15 dwt.; what would be the price of it at the above rate?

24. The number of acres growing hops in England, in 1850, was 43127: the duty on these was at the rate of £9 17*s*. per acre: required the whole amount of duty?
25. In the year ending Jan. 5, 1851, 2623656 gallons of spirits were imported into England from Scotland, and 828138 gallons from Ireland; the duty paid on the transfer was 7*s*. 10*d*. per gallon: what was the whole amount of duty?
26. The prices paid at St. Thomas's Hospital, London, for beef and mutton during the year 1850, were for the former 2*s*. 8*d*. per stone, and for the latter 3*s*. 4*d*. per stone: how much was paid for 13 cwt. of each?
27. The average pay of the crew of a foreign-going trading vessel is as follows: captain, £10 per month; mate, £5; second mate, £2 15*s*.; carpenter, £4; able seaman, £2 5*s*. The number of able seamen is 2 to every hundred tons registered: what is the pay of the company of a ship of 800 tons for 107 days, allowing 30 days to the month?
28. Every person in Great Britain who receives an annual income of £150, or more, must pay an income-tax of 7*d*. in the £: what amount of tax must a person pay whose income is £2625?
29. The dearest year for provisions ever known in England was the year 1813; the contract price of butchers' meat paid by Government for the supply of Greenwich Hospital was then £4 5*s*. per cwt.: as the average charge for beef and mutton in 1850 was 3*s*. per stone, calculate the reduction in price per cwt. and per stone.
30. The quantity of wheat sold in the United Kingdom in 1850 was 4688246 quarters; of barley, 2235271 quarters; and of oats 866082 quarters: the average prices for the year were, wheat, 40*s*. 3*d*. per quarter; barley, 23*s*. 5*d*. per quarter; and oats, 16*s*. 5*d*. per quarter: what sum was received for the whole?

PROPORTION.

(67.) Four quantities are said to be in proportion when the first contains the second as many times and parts of a time, as the third contains the fourth, or when the complete quotient of the first, divided by the second, is the same abstract number as the complete quotient of the third divided by the fourth.*

(68.) The complete quotient arising from dividing one quantity by another of the same kind is called the *ratio* of the former to the latter: thus the ratio of 6 to 3 is 2; the ratio of 6 to 2 is 3; the ratio of 7 to 4 is $1\frac{3}{4}$; and so on. *Ratio* is thus only another name for *quotient:* the first *term* of the ratio, that is the dividend, is called the *antecedent;* and the other term, that is the divisor, is called the *consequent.* When, therefore, four quantities are said to be in proportion, or to form a proportion, you are merely to understand, that the ratio of the first to the second is the same as the ratio of the third to the fourth; in other words, that if you were to divide the first by the second, you would get the same complete quotient, as if you were to divide the third by the fourth. For example, the four numbers, 12, 2, 18, 3, form a proportion, because the ratio of 12 to 2, namely 6, is the same as the ratio of 18 to 3. Instead of saying in words that these four numbers are in proportion, it would be sufficient to write them in a row, with dots between them, as follows: 12 : 2 :: 18 : 3, which would be read, 12 *is to* 2, *as* 18 *to* 3; or, *as* 12 *is to* 2, *so is* 18 *to* 3. You see that the two dots which separate the *terms* of each ratio, differ from the sign for division ($\div$) only by the little mark between them; and, in fact, the notation just employed is only the same as $12 \div 2 = 18 \div 3$, and you may always regard a proportion in

* The learner will observe, that I say here *the same abstract number*, because a *ratio*, implying the relation of one quantity to another with respect to the magnitude of them, can exist only between quantities of the same kind. It has already been seen (page 62), that *division* and *quotient*, unlike *multiplication* and *product*, are terms that are used in different senses: we are said to *divide* a concrete quantity by 4, when we take the fourth part of that quantity; which fourth part we are accustomed to call the *quotient*. If division were restricted to mean the finding of how many times one quantity is contained in another, the operation just noticed would be excluded; the term *quotient* is to be understood in this restricted sense when used for *ratio*.

this light; indeed, you will often find the term *proportion* to be briefly defined as "*an equality of ratios*," or *quotients*.

(69.) I think, from this explanation, you clearly see the meaning of *ratio*, as applied to *two* numbers, and of *proportion*, as applied to *four;* that if two numbers were proposed, you could tell the ratio of the first to the second; and that if four were proposed, you could find out whether they were in proportion or not: thus, if the two numbers 24 and 6 were proposed, you would know that the ratio of the first to the second is 4; that the ratio of 8 to 2 is also 4; and you would thus infer, that the four numbers, 24, 6, 8, 2, are in proportion; or, that 24 : 6 :: 8 : 2. Again; the ratio of 17 to 4 is $3\frac{3}{4}$, so that the four numbers, 24, 6, 17, 4, are *not* in proportion; the ratio of the first to the second being greater than the ratio of the third to the fourth. In like manner, the ratio of 6 to 8 is $\frac{6}{8}$ or $\frac{3}{4}$; and the ratio of 15 to 20 is $\frac{15}{20}$ or $\frac{3}{4}$; therefore, 6 : 8 :: 15 : 20; also, 6 : 8 :: 3 : 4. When you have once got an antecedent and consequent, that is, the two terms of a ratio, you can easily get another antecedent and consequent, that is, two other terms, in *the same ratio;* and can thus form a proportion, having the given antecedent and consequent for the first two terms of it: suppose, for instance, you had 6 and 8 for antecedent and consequent, and you wanted a proportion in which 6 and 8 should stand first; you have only to remember, that the ratio of 6 to 8 is expressed by the fraction $\frac{6}{8}$, and that numerator and denominator of a fraction may be multiplied or divided by any number you please, in order to get as many other suitable pairs of numbers as you choose: thus, since $\frac{3}{4} = \frac{6}{8} = \frac{12}{16} = \frac{36}{48} = \frac{30}{40}$, &c., you infer at once that all the following are proportions, namely, 6 : 8 :: 3 : 4; 6 : 8 :: 12 : 16; 6 : 8 :: 30 : 40; 12 : 16 :: 36 : 48; and so on. And whenever you have two fractions equal to one another, you may always convert the *equality* into a *proportion*, and say, the numerator of the first *is to* its denominator, *as the* numerator of the second *to* its denominator; or, instead of saying numerator *is to* denominator, you may, if you please, say denominator *is to* numerator, because if two fractions are equal, the equality remains, though we make the numerator and denominator of each change places; and it is because of this, that in any proportion we may make the antecedent and consequent of each ratio change places: thus, since it is true, that 3 : 7 :: 9 : 21, it is also true, that 7 : 3 :: 21 : 9; and

since $3 : 7 :: 1\frac{1}{2} : 3\frac{1}{2}$, so also $7 : 3 :: 3\frac{1}{2} : 1\frac{1}{2}$; you, of course, see that $\frac{3}{7} = \frac{1\frac{1}{2}}{3\frac{1}{2}}$, since the latter fraction arises from dividing numerator and denominator of the former by 2. From all this, it appears, that *a proportion may always be converted into a pair of equal fractions, and a pair of equal fractions into a proportion:* the word *fraction* being here used in its widest sense, and applying to every quantity expressed in a fractional form. By thus replacing a proportion by two equal fractions, many properties of proportional quantities may be easily deduced; I shall here notice only two. Since

$$\frac{\text{first term}}{\text{second term}} = \frac{\text{third term}}{\text{fourth term}}, \text{ or } \frac{\text{fourth term}}{\text{third term}} = \frac{\text{second term}}{\text{first term}}$$

it follows that $\text{fourth term} = \frac{\text{second term}}{\text{first term}} \times \text{third term}$,

a property which indicates to us how we are to find the fourth term of a proportion, when only the first, second, and third terms of it are given. Of these three terms, the first two, that is, the first antecedent and consequent, must always be quantities of *the same kind;* I mean, that both of them must be either abstract numbers, or both of them concrete quantities, belonging to one class of quantities: if the first term, for instance, be *money*, the second must be *money;* if the first be *time*, the second must be *time;* and so on. You must see the necessity of this from the very nature of *ratio*, as defined at the beginning of this section; it is the *number of times* the antecedent, or first term of the ratio, contains the consequent, or second term; and therefore *ratio is always an abstract number.*

(70.) This you must be careful to bear in mind, else you will run the risk, in finding the *fourth proportional*, as it is called, to three given concrete quantities, of committing an absurdity at every step of your work, though you may bring out the correct result. I have shown to you above, how the fourth term, or fourth proportional, may always be found in an unobjectionable manner; you there see that you are to divide the second term by the first, and to multiply the third term by the resulting *abstract number:* you will thus get the sought fourth term, and always in the *same denomination as the third term.* If the terms of both the equal ratios constituting the proportion, or if the terms of only one of

these equal ratios, were abstract numbers, then we might deduce from the general property above, this other particular property, namely,

first term × fourth term = second term × third term;

or, that *the product of the extremes is equal to that of the means;* a statement which, you see, would be absurd, if the first and fourth, or the second and third, were *both* concrete quantities.

(71.) If the first and second of the three given terms of a proportion were always abstract numbers, then, from what has just been shown, we could find the fourth term by multiplying the second and third together, and dividing by the first, or cutting the product up into as many equal parts as there are units in the first term: and this would, in general, be an easier way of getting at the fourth term, than by dividing the second term by the first, and then multiplying the third term by the quotient, as directed above. Now I wish you particularly to observe, that we may always proceed in the latter easier way, even when the first and second terms are *concrete.* You know you cannot divide one concrete quantity by another, till both are brought to the same denomination: having done this, your quotient is an abstract number;—the very same abstract number that you would get if the common denomination of dividend and divisor were wholly disregarded, and the division performed on the abstract numbers simply. In the case, therefore, of the first and second terms of a proportion being concrete quantities, all you have to do is to reduce these quantities to the *same denomination;* then, leaving *denomination* altogether out of consideration, to employ only the resulting *abstract numbers,* which you see may always be put instead of the quantities themselves: for the *ratio,* or *quotient,* remains unaltered, whether the denomination (or the concrete unit) be preserved or suppressed.

(72.) What is called the Rule of Three, or the Rule of Simple Proportion, is merely the method of finding the fourth term of a proportion when the first three are given. In most of the questions coming under this rule, the three given quantities do not occur in the order in which you would write them as three of the four terms of the proportion: in the *question* you will usually find that the quantities which stand first and third must stand first and second, or second and first, when arranged as terms of a *proportion;* the following, for

instance, is such a question: If 3 lbs. of sugar cost $16\frac{1}{2}d.$, what will 10 lbs. cost? Here, it is plain that the 3 lbs. and the 10 lbs. must bear the same relation to one another, that is, must have the same ratio, as the price of the 3 lbs. to the price of the 10 lbs. Hence the *stating* of the question, as it is called, would be: 3 lbs. : 10 lbs. :: $16\frac{1}{2}d.$; or, suppressing the common denomination, lbs., of the terms of the first ratio, 3 : 10 :: $16\frac{1}{2}d.$ As the third term here is *pence*, the wanting fourth term, which is to complete the proportion, must also be *pence*. It is found agreeably to what is said above, by multiplying the $16\frac{1}{2}d.$ by the 10, and dividing the product by the 3; that is, it is $\frac{16\frac{1}{2}d. \times 10}{3} = 55d.$ The general rule for all questions of this kind is as follows:—

(73.) Rule of Three, or Simple Proportion

Rule 1. Write down the three given quantities as the first three terms of a proportion, taking care that the third term is a quantity of *the same kind* as the required fourth term; and according as this fourth term is to be greater or less than the third, so let the second term be greater or less than the first.

2. Having thus *stated* the question, reduce the first and second terms to the same denomination, if they are not already the same as well in denomination as in kind; and then, disregarding the *denomination*, consider the first and second terms to be abstract numbers, divide them by any number that will *obviously* divide both, and use the quotients instead; which, as they will be smaller numbers, may in general be more easily worked with.

3. Multiply now the third term by the second, and divide the product by the first term, and the quotient will be the answer, or fourth term of the proportion; and it will be in that denomination, whatever it be, in which the third term was used.

(74.) Note 1. It may be proper here to remind you, that the first and second terms of a proportion may be regarded as the two terms of a *fraction;* and the third and fourth terms as the two terms of another fraction equal to the former. It is by viewing an antecedent and consequent in this light, that we perceive our right to divide both by any number we please; and further, that according as the first term of a

proportion is greater or less than the second, so must the third be greater or less than the fourth. This fact will be a sure guide to the correct stating of a Rule-of-Three question. A little attention to the question will always enable you to see whether the fourth term you seek ought to be greater or less than the third, which is given; so that you need never fall into the mistake of putting that term first which ought to be second.

NOTE 2. As the fourth term or answer to the question is always got by multiplying the third term by the second, and dividing the product by the first, using the first and second as abstract numbers, you may regard the work as indicated by a *fraction*, of which the numerator is the product of the third term by the second, and the denominator the first term; and as you may divide the numerator and denominator of a fraction by any number you please, you may obviously divide not only the first and second, but also the first and third of the terms, by any number that will really simplify them; and may work with the simplified results instead of with the terms themselves; but you must be careful not to take this liberty with the *second* and *third* terms. An example or two will make you familiar with the process.

Ex. 1. If 8 articles cost £21 4*s*., how much will 26 cost?

Here the fourth term of the proportion, that is, the answer to the question, must be *money;* we therefore make money the *third* term; and as the required fourth term must evidently be greater than the third, we take care that the second term is greater than the first; and therefore state the question as in the margin. We now look at the first and second terms, and readily see that both will divide by 2; we therefore replace 8 : 26 by 4 : 13; and as £21 4*s*. is not easily multiplied by 13, we reduce the compound quantity to the denomination shillings before using it; we then multiply the shillings thus obtained by 13, and divide by the 4, as the rule directs. We thus find that the fourth

```
                 £.  s.    £.  s.
8 : 26  ::  21   4  :  68  18
4 : 13       20
            ----
             424
              13
            ----
            1272
            424
            ----
          4)5512
            ----
        2,0)137,8s.
            ----
     Ans.  £68 18s.
```

term of the proportion, or the answer to the question, is £68 18*s.*; and we make the original stating complete by inserting the fourth term, now found, in its proper place. The work would have been a little easier if, instead of first multiplying by 13 and then dividing by 4, we had first divided by 4 and then multiplied by 13; but if, after having replaced 8 : 26 by 4 : 13, we had divided the first and third terms by 4, the work would have been easier still; for the stating would then have been 1 : 13 :: £5 6*s.*, and we should have got the answer at once, by multiplying £5 6*s.* by 13, which of course is easily done; the first term being 1, no *division* is performed.

2. If 19 cwt. of sugar cost £57, how much may be bought for £111?

Here the answer must be *weight;* we therefore put the given weight for the third term; and as the required weight is greater than the given weight, of the other two given quantities we take care to put the greater second; the stating is therefore as in the margin: and as we easily see, that instead of 57 : 111 we may put 19 : 37, we accordingly use this simplification. As 19 times 37 is the same as 37 times 19, we find the product in the former way for convenience. But you see there was not, in reality, any occasion for multiplying at all; for dividing the first and third by 19, the stating becomes simply 1 : 37 :: 1 cwt. : 37 cwt.; so that the answer is got at once.

£.   £.   cwt. cwt.
57 : 111 :: 19 : 37
19 : 37
19
———
333
37
———
19)703(37 cwt. Ans.
57
———
133
133
———

From the remarks appended to the two examples here solved, you cannot but perceive the advantage of a little preliminary examination of your *stating* before you begin to apply the rule. The work given at length in the margin is intended more for your *avoidance* than for your *guidance* in similar cases. A little thought and ingenuity on your part will often do more for you than all the *rules* of arithmetic.

3. What is the yearly rent of 47 ac. 3 roo. 21 per. at £1 4*s.* 6*d.* per acre?

per.	ac.	roo.	per.	£.	s.	d.		£.	s.	d.
160 :	47	3	21	1	4	6	:	58	13	$1\frac{7}{80}$
80	4				12	3				
					12					

191
40

7661
147
———
53627
30644
7661
———
8,0)11261,6,7
———
12) $14077\frac{7}{80}d$
———
2,0) 117,3 1*d.*
———
£ 58 13*s.* $1\frac{7}{80}d.$ Ans.

147*d.*

Here the money must be the third term, 1 acre or 160 per. the first, and the other quantity the second. This latter must be reduced to perches, in order that the first two terms may be of one common denomination, the common denomination (perches) being disregarded after the reduction. Previously, however, we may simplify the first and third terms, by taking only the half of each. This done, the money, for convenience of multiplication, is reduced to pence, and the work stands as in the margin. The yearly rent is thus found to be £58 13*s.* 1*d.*; the fraction, which is only about one-third of a farthing, being disregarded.

4. If 37 workmen can complete a piece of work in 241 days, in how many days would 57 men finish it?

As the answer is to be days, 241 days must be the third term; and since the greater the number of workmen the less the time, the answer must be less than 241 days; therefore the first two terms must be 57 : 37; and the work as in the margin, the number of days being found to be $156\frac{25}{57}$ days.

days. days.
57 : 37 :: 241 : $156\frac{25}{57}$
37
———
1687
723
———
57)8917($156\frac{25}{57}$
57
———
321
285
———
367
342
———
25

5. The following is an example in fractions: If $1\frac{1}{4}$ cwt. of lead cost $36\frac{3}{8}s.$ what will $3\frac{2}{5}$ cwt. cost?

$$1\tfrac{1}{4} : 3\tfrac{2}{5} :: 36\tfrac{3}{8}s. : 98\tfrac{47}{50}s.$$
$$10 : 3\tfrac{2}{5} :: 291s.$$
$$\frac{291 \times 17}{5 \times 10}s. = 98\tfrac{47}{50}s. = £4\ 18s.\ 11\tfrac{1}{4}d.$$

Here, if we multiply the first and third terms by 8, we shall get rid of the fractions in those terms, and the stating will then be as in the margin. The second term, reduced to an improper fraction, is $\frac{17}{5}$; hence, multiplying the 291*s.* by this, and dividing by the 10, we find the answer to be $98\frac{47}{50}s.$

Exercises.

1. If 17 lbs. cost 11*s.* 7*d.*, what will 23 lbs. cost?
2. If 9 lbs. of tea cost £1 18*s.* 6*d.*, what will 1 cwt. cost?
3. What is the price of six cheeses, each weighing $52\frac{3}{4}$ lbs., at $5\frac{3}{4}d.$ per lb.?
4. If 28 persons reap a harvest in 36 days, how many will be required to reap it in 21 days?
5. If a garrison of 1000 soldiers have provisions for 9 months, how many must be dismissed in order that the provisions may last 15 months?
6. A besieged garrison has 5 months' provisions, allowing 12 oz. a day for each man, but finding that it must hold out for 9 months, how much must each man have per day to make the provisions last?
7. What must be paid for 1 cwt. 3 qr. 17 lb. of wool, at 7*s.* 4*d.* per stone of 14 lbs.?
8. If 1787 cwt. 2 qr. of lead cost £907 10*s.*, what is that per fother of $19\frac{1}{2}$ cwt.?
9. If the entire rental of a parish amount to £2500, and a poor-rate of £112 2*s.* is to be raised, what must a person pay whose rental is £525?
10. If five-eighths of a ship be worth £525, what is the value of three-fourths of three-sevenths?
11. In a single mass, weighing 3 cwt., found in July 1851, at about 50 miles from Bathurst, in Australia, there were discovered to be 106 lbs. of gold; what would this fetch, at the rate of £3 6*s.* 8*d.* per ounce?
12. From Sept. 29, 1850, to Sept. 27, 1851, there died of the population of London, within the walls of the city, 2978 persons, giving about 23 deaths for every 1000 persons living at the latter date: what was the amount of population of that part of London in Sept. 1851?

13. From 1 lb. of standard gold, $44\frac{1}{2}$ guineas used to be coined: how many sovereigns are now coined out of the same weight?
14. 1 lb. avoirdupois is heavier than 1 lb. troy; for 144 lbs. avoirdupois are equal to 175 lbs. troy: what is the troy weight of 1 lb. avoirdupois?
15. The imperial gallon contains 12 lbs. 1 oz. 16 dwt. 16 gr. troy weight of distilled water: how many pounds avoirdupois does it contain?
16. At what time between 7 and 8 o'clock are the hour and minute hands exactly in opposition, or in the same straight line? *
17. At what time between 5 and 6 o'clock are the hour and minute hands exactly together?
18. Eleven Irish miles are equal to 14 English miles: what is the length in English miles of a road which measures 57 Irish miles?
19. A ream of paper contains 20 quires, and a quire contains 24 sheets: what would be the cost for paper for 2500 copies of a book containing $7\frac{3}{4}$ sheets, at 15*s.* 6*d.* per ream?
20. The average price of wheat for the year 1830 was 64*s.* 3*d.* per quarter; and for the year 1850 it was 40*s.* 3*d.*: the sixpenny loaf in the latter year weighed 4 lb., what did it weigh in 1830?
21. The shadow of a cloud was observed to move 36 yards in 5 seconds: what was the hourly motion of the wind?
22. If a person pays £22 7*s.* 5*d.* for income-tax, at the rate of 7*d.* in the £, what is his income?
23. There are 18 dwt. of alloy in 1 lb. of standard silver; this 1 lb. is coined into 66 shillings: how much pure silver is there in 20*s.*?
24. What was the weight of the £275000 taken in silver coin at the doors of the Great Exhibition of 1851, in tons, cwts., &c. *avoirdupois*?

* In order to work this exercise, the learner must remember that the minute-hand moves 12 times as fast as the hour-hand, so that while the short-hand goes over any space, the long-hand *gains upon it* 11 times that space. Now, it is plain, that under the conditions of the question, the *gain* of the long-hand will be 1 hour-space, the space from XII to I; and it is required to determine what advance the short-hand must make to allow of this gain, the space gone over by the short-hand being *always* to the *gain* of the long-hand as 1 to 11. By the same considerations the next question may be easily answered.

[For additional exercises in this rule, the learner may take the examples already given under the head of Practice.]

(75.) Double Rule of Three, or Compound Proportion.

The Double Rule of Three is so called, because it implies, at least, *two* single Rule-of-Three statings, every question coming under this rule being resolvable into, at least, *two* questions, each of which may be worked by the single Rule of Three. This will be best understood by an example.

Suppose 6 men can mow 9 acres of grass in 4 days, how many men will be required to mow 27 acres in 3 days?

This question may be divided into two, thus: 1st. If 6 men can mow 9 acres in 4 days, how many men can mow *the same* in 3 days? Here, 3 : 4 :: 6 men : 8 men. Consequently, 8 men can mow the 9 acres in 3 days.

2nd. If 8 men can mow 9 acres in 3 days, how many will be required to mow 27 acres in *the same time?* Here, 9 : 27 :: 8 men : 24 men. It is plain, therefore, that the answer to the question is 24 men.

In the first of these proportions, the fourth term is got by dividing the third term (6 men) by the ratio 3 : 4; that is, by multiplying by the fraction $\frac{4}{3}$: in the second, the fourth term is got by dividing what has just been found (8 men) by the ratio 9 : 27; that is, by multiplying by the fraction $\frac{27}{9}$. The answer to the original question is, therefore, obtained by dividing the third term in the *first stating* by the *product* of the ratios 3 : 4 and 9 : 27, that is, by $3 \times 9 : 4 \times 27$, or, by multiplying by $\frac{4 \times 27}{3 \times 9}$. The product of two ratios is called their *compound ratio*, and this is why the Double Rule of Three is called also *compound proportion;* and examples in it are usually worked, not by working with the several ratios singly, as above, but by taking the compound ratio, at once. In this way, the stating of the above question would have been written as follows.

$\left.\begin{matrix} 3 : 4 \\ 9 : 27 \end{matrix}\right\}$ :: 6 men; that is $3 \times 9 : 4 \times 27$ :: 6 men; the fourth term of which proportion is $\frac{4 \times 27 \times 6}{3 \times 9}$ men $= 4 \times 3 \times 2 = 24$ men. There are sometimes more than two simple

ratios to be compounded: thus, in the present example, it might have been a condition, that the 4 days occupied in mowing the 9 acres were 8 hours long, and that the days occupied in mowing the 27 acres were to be 12 hours long; then, as the longer days would require fewer men, a third ratio would have been 12 : 8; so that only $\frac{8}{12}$, or $\frac{2}{3}$, of 24 men would have sufficed; that is, 16 men.

I think you will now be prepared for the following general rule for all questions of this kind.

Rule 1. As in the Single Rule of Three, put, for the third term, that one of the given quantities which is of the same kind as the quantity sought.

2. Then, selecting any pair of the remaining quantities, like in kind, complete the stating, just as if these three were the only quantities given in the question, disregarding all the others.

3. In like manner, take another pair, like in kind, from the given quantities, and place them under the former pair, and so on, till all the pairs are used; two dots, to signify ratio, being put between the terms of each pair; and these terms being arranged, as to first and second, just as you would arrange them if they and the third term were the only quantities concerned,

4. Multiply the third term by the product of all the consequents of these ratios, and divide the result by the product of all the antecedents: the quotient will be the answer.

Note. The terms of each of the given ratios, together with the common third term, may be reduced, when possible, to smaller numbers, just as if they were the three terms of a simple Rule-of-Three stating; or the multiplications and divisions, implied in the rule, may be *indicated* by the signs for these operations; and factors, common to multipliers and divisors, struck out before the operations are actually performed; the factors that may be struck out, will often be discovered, by merely inspecting the simple ratios as they stand: thus, in the example worked above, the stating $\left.\begin{array}{l}3 : 4\\ 9 : 27\end{array}\right\} :: 6 :$ may be replaced by $\left.\begin{array}{l}1 : 4\\ 1 : 3\end{array}\right\} :: 2 :$; therefore, $1 : 12 :: 2 : 24$. It is plain, that a factor common to *any* antecedent and a consequent, may always be struck out, since the consequents are all so many multipliers, and the antecedents so many divisors.

You will, of course, understand, that what is called the common third

term in a double rule-of-three stating, is not the third term common to each of the single rule-of-three statings, into which the former may be decomposed; although in arranging the terms of each pair of component ratios, you are guided as to whether the greater or the less should be placed first, just as if this third term were that of the corresponding single rule-of-three stating; you may easily convince yourself that you can never be misled into a wrong disposal of the first two terms by thus consulting the third term. In the second single rule-of-three stating, the true third term is that which actually appears multiplied by a fraction, so that the true fourth term would be the fourth term corresponding to the third which actually appears multiplied by the same fraction; and it is plain, that according as one quantity is greater or less than another, so will any fraction of the former be greater or less than that fraction of the latter; consequently, in inquiring whether the third or fourth term be the greater, the fractional multipliers may be disregarded.

Ex. 1. If 12 horses plough 11 acres in 5 days, how many horses would plough 33 acres in 18 days?

$$\left.\begin{array}{r}11 : 33\\ 18 : 5\end{array}\right\} :: \overset{hor.}{12} :$$

or

$$\left.\begin{array}{r}1 : 3\\ 3 : 5\end{array}\right\} :: \overset{hor.}{2} :$$

or

$$\left.\begin{array}{r}1 : 1\\ 1 : 5\end{array}\right\} :: \overset{hor.}{2} : \overset{hor.}{10}.\quad \text{Ans.}$$

Here, as the answer is to give the number of *horses*, we put horses for the common third term; and, disregarding days, just as we should do if the number of days were the same in both cases, we consider merely the acres ploughed; and as more horses are required for 33 acres than for 11, the first ratio is 11 : 33. Again; returning to the question, we now disregard acres, and consider only days; and as fewer horses are required for 18 days than for 5 days, the second ratio is 18 : 5; hence the work stands as in the margin. The common factor 11 is struck out of 11 : 33, and the common factor 6 out of the first and third terms, 18 and 12; and, lastly, the common factor 3 is struck out of the consequent in the first ratio, and the antecedent in the second: with these simplifications, the work is reduced merely to the multiplication of 2 by 5.

You will, of course, observe that the *denomination*, common to the first and second terms of each of the given ratios, is wholly disregarded in the work, the abstract numbers alone being used.

2. If 15 men, working 12 hours a day, reap 60 acres in 16 days, in what time would 20 women, working 10 hours a day, reap 98 acres; 7 men being able to do as much work as 8 women in the same time?

As the answer here is to be days, we put the given number of days for our third term; then regarding the number of workers only, just as if all the other conditions were the same, our first ratio is 20 : 15, because 20 workers is to 15 workers as 16 days to a *less* number of days; next considering only the number of hours, as if these alone varied, the second ratio is 10 : 12, as there must be a greater number of 10-hour days than of 12-hour days, and our answer is to be in 10-hour days. Again, taking the acres only into account, the next ratio is 60 : 98; and, lastly, the ratio of a man's time of doing any amount of work to a woman's is 7 : 8. Hence the complete stating is as in the margin. The factor 15, as also the factor 4 in the 12, may be struck out from the numerator, and at the same time the factor 60 from the denominator. Again the factor 7 may be struck out from 98 in the numerator, and the same factor suppressed in the denominator; we shall thus have the reduced form, $\frac{3\times 14\times 8\times 16}{20\times 10}$. Lastly, striking out the factor 4 from the 8 and the 20, and the factor 2 from the 14 and the 10, we have finally the fraction $\frac{3\times 7\times 16}{5\times 5}=26\frac{23}{25}$. It usually occupies less space to work our way to the final result in this manner, than to reach it by successive simplifications of the original stating, as in Ex. 1. That example treated in this way, would give at first the fraction $\frac{33\times 5\times 12}{11\times 18}$; which, by striking out common factors, becomes simply 5×2 or 10.

$$\left.\begin{array}{r} 20 : 15 \\ 10 : 12 \\ 60 : 98 \\ 7 : 8 \end{array}\right\} :: \begin{array}{c}\text{days. days.}\\ 16 : 26\frac{23}{25}\end{array}$$

$$\frac{15\times 12\times 98\times 8\times 16}{20\times 10\times 60\times 7}=\frac{3\times 7\times 16}{5\times 5}=26\tfrac{23}{25}\text{ days. Ans.}$$

Exercises.

1. If 14 horses eat 56 bushels of oats in 16 days, how many horses will 120 bushels keep for 24 days?
2. If a person walking 12 hours a day travel 250 miles in 9 days, in how many days of 10 hours each could he walk 400 miles, at the same rate?
3. If 12*s*. be paid for the carriage of 2 cwt. 3 qr. 192 miles, how much should be paid for the carriage of 8 cwt. 1 qr. 128 miles?

4. If 3000 copies of a book of 11 sheets require 66 reams of paper, how much paper will be required for 5000 copies of a book of $12\frac{1}{2}$ sheets?
5. If 24 men can finish a piece of work in 36 days of 12 hours each, in what time can 30 men do it when the working days are only 8 hours long?
6. If 939 soldiers consume 351 quarters of wheat in 168 days, how long will 1404 quarters last 11268 soldiers?
7. If the sixpenny loaf weigh 32 oz. 8 dwt. when wheat is 60*s.* per quarter, what should the eightpenny loaf weigh when wheat is 54*s.* per quarter?
8. If a family of 13 persons spend £64 in butcher's meat, in 8 months, when meat is 6*d.* per lb., how much money, at the same rate, should a family of 12 persons spend in 9 months, when meat is $6\frac{1}{2}$*d.* per lb.?
9. If the rent of a farm of 13 ac. 1 roo. $11\frac{1}{2}$ per. be £50 8*s.* 9*d.*, what should be the rent of another in the neighbourhood, containing 8 ac. 3 roo. 22 per., if 6 acres of the latter be worth 7 of the former?
10. If £7 10*s.* be the wages of 15 men, who work 10 hours a day for 6 days, what ought to be the wages of 12 men who work 9 hours a day for $18\frac{1}{2}$ days?

(76.) DECIMALS.

I HAVE already explained, at the commencement of this rudimentary treatise, that our notation for integers, or whole numbers, is the *decimal* notation, inasmuch as the value of any figure of a number is *ten* times as much as it would be if that figure were removed one place to the right, so that in writing the figures of a number in the usual way, from left to right, every figure we put down is, in value, only the *tenth part* of what it would be, if it were one place less in advance. Now, the whole number becomes completed as soon as we reach in this way the place of *units;* but there is no reason why the decimal notation should not be extended to the right *beyond* the place of units, still considering the value of each figure we write down, to be only the *tenth part* of what it would have been, if written in the immediately preceding place. In this way, the first figure written after *units*, would be *tenths;* the next figure, *hundredths;* the next,

thousandths; and so on: and, to prevent confusion, we should only have to put some mark of separation between the *units* and these *fractional parts.* This extension of the decimal notation is what we now have to consider. The mark employed to separate the decimal *integers* from the decimal *fractions* is simply a dot: thus, 234·625, means 234, with 6 *tenths*, 2 *hundredths*, and 5 *thousandths;* it is therefore the same as $234+\frac{6}{10}+\frac{2}{100}+\frac{5}{1000}$. Each figure to the right of the *decimal point* is thus a *fraction* of known denominator, although that denominator does not appear; and such fractions are properly called *decimal fractions*, on account of the regular ten-fold increase of the denominators: for brevity, however, they are usually called simply *decimals.*

This extension of the decimal notation is so natural and obvious, that you can have no difficulty in understanding it; and as soon as an example of it is placed before you, you can as readily pronounce upon the *value* of a figure to the right of the decimal point, as you can pronounce upon the value of a figure to the left: the place next to units, on the *left*, is *tens;* the place next to units, on the *right*, is *tenths;* the place next to *tens*, on the left, is *hundreds;* the place next to *tenths*, on the right, is *hundredths;* and so on; thus, in the mixed number, 1234·5678, you could as readily tell the value of the 6 as of the 2, each of which is in a *third* place, the 4 being in the first, or units place: the value of the 2 is 200, or 2 *hundreds;* the value of the 6 is $\frac{6}{100}$, or 6 *hundredths:* in like manner, the value of the 3 is 30, or 3 *tens*, and of the 5, $\frac{5}{10}$, or 5 *tenths;* the value of the 1 is 1000; and of the 7, $\frac{7}{1000}$; and, lastly, the value of the 8 is $\frac{8}{10000}$. It is very convenient to be able to express, in this way, decimal fractions without the incumbrance of denominators; and the more so, since, as you will presently see, *all* fractions may be converted into *decimal* fractions. From what has just been said, you see, that in order to express a decimal as a common fraction (sometimes called a *vulgar* fraction), you have only to write the figures of the decimal for numerator, and for denominator, to put 1, followed by as many zeros as will mark the place of the last decimal figure from the decimal point: thus, in the instance before given, it was seen, that $\cdot625=\frac{6}{10}+\frac{2}{100}+\frac{5}{1000}$, which is obviously $\frac{625}{1000}$, the three zeros corresponding to the *third* place of the last decimal figure, 5. In like manner, $\cdot2438=\frac{2}{10}+\frac{4}{100}+\frac{3}{1000}+\frac{8}{10000}=\frac{2438}{10000}$; $\cdot0342=\frac{3}{100}+\frac{4}{1000}+\frac{2}{10000}=\frac{342}{10000}$; $\cdot0036=$

$\frac{3}{1000}+\frac{6}{10000}=\frac{36}{10000}$; and so on; the number of zeros being always equal to the number of decimal places. There is one thing here that will no doubt occur to you; it is this; that although zeros immediately *after* the decimal point, that is, *before* the figures, materially affect the value of those figures; yet, that zeros *after* them, have no effect at all, and are quite useless: thus, 234·625 is the same as 234·625000, &c.; and ·0036 is the same as ·0036000, &c.; zeros intermediate *between* decimal figures, have, of course, the effect of pushing the figures which follow them, lower down in the scale: thus, ·62 is $\frac{6}{10}+\frac{2}{100}$, but ·602 is $\frac{6}{10}+\frac{2}{1000}=\frac{602}{1000}$, and ·6002 is $\frac{6}{10}+\frac{2}{10000}=\frac{6002}{10000}$.

(77.) The removal of the decimal point one place to the right, is equivalent to multiplying the decimal by 10; the removal of it two places to the right, is equivalent to multiplying the decimal by 100; and so on: thus, $\cdot 2438\times 10=2\cdot 438$, where each figure is 10 times what it was before; $\cdot 2438\times 100=24\cdot 38$; $\cdot 2438\times 1000=243\cdot 8$, &c.: and the removal of the point in the other direction, is equivalent to dividing the decimal by 10, 100, &c.: thus, $\cdot 2438\div 10=\cdot 02438$; $\cdot 2438\div 100=\cdot 002438$; $243\cdot 8\div 100=2\cdot 438$, &c. &c. All this is plain from the very notation of decimals.

(78.) I shall now give you an example or two of converting decimals into fractions: 1. $17\cdot 5=17\frac{5}{10}=17\frac{1}{2}$. 2. $21\cdot 25=21\frac{25}{100}=21\frac{1}{4}$. 3. $146\cdot 75=146\frac{75}{100}=146\frac{3}{4}$: you thus see, that $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, expressed in decimals, are ·5, ·25, ·75. 4. $6\cdot 14\times 10=61\cdot 4=61\frac{4}{10}=61\frac{2}{5}$. 5. $3\cdot 135\times 100=313\cdot 5=313\frac{1}{2}$. 6. $2\cdot 76\div 100=\cdot 0276=\frac{276}{10000}=\frac{69}{2500}$.

It is further obvious, from the principles of the decimal notation, that when that notation is exchanged for the fractional, as in these examples, by writing the decimals without the point, and putting underneath, for denominator, unity, followed by as many zeros as there are decimal places, we may prefix to the numerator whatever whole number may have preceded the decimal: thus, $146\cdot 75=146\frac{75}{100}=\frac{14675}{100}$; $17\cdot 5=\frac{175}{10}$; $21\cdot 25=\frac{2125}{100}$; $326\cdot 047=\frac{326047}{1000}$; and so on. This is evidently only the same as reducing a mixed number to an improper fraction.

In writing decimals, you must be careful to put the decimal point against the *upper* part of the figures, not against the lower. When figures are separated by a point even with the lower part of the figures, the *multiplication* of the figures separated is understood, the point in that position standing in the place

of the sign $\times$: thus, 3.7 is the same as 3×7, or 21, while 3·7, is 3 and 7 *tenths*; or, as it is usually read, 3 *decimal* 7, or 3 *point* 7: in like manner, 24.36, means 24×36, while 24·36, is 24 *decimal* 36, or 24 *point* 36; that is, 24 and 36 *hundredths*, and so on. It is necessary that you should keep in mind this new sign for multiplication, as it is very frequently used; so that, whenever you meet with such an expression as 2.3.7.4, you may know that it means $2\times 3\times 7\times 4$. There is another useful little sign of abbreviation, which it is high time you should be made acquainted with: it is the sign $\therefore$, which is the mark, not for an *operation*, but for a *word*—the word *therefore* (or, *consequently*), —a word of such frequent occurrence, in numerical inquiries, as to render a short sign for it very acceptable. Henceforth, I shall use this sign $\therefore$ for *therefore*, rather than introduce the *word* in the midst of arithmetical work.

(79.) *To reduce a Common Fraction to a Decimal.*

Rule 1. Annex a zero to the numerator, which then take for a dividend, the denominator being the divisor: if the dividend be sufficiently large, find the first figure of the quotient; but if it be too small to give a significant first figure, put zero for the first figure, and annex another zero to the dividend; if this be still too small, put a second zero in the quotient, and annex another zero to the dividend; and so on, till the dividend be large enough to give a significant figure in the quotient.

2. If there be a remainder, conceive another zero to be annexed; and continue the division, still annexing a zero to every remainder, till the work terminates of itself, or till the quotient has been carried to as many places as may be required: this quotient, with the decimal point before it, will be the value of the proposed fraction in decimals.

Note. It will be enough if we *imagine* the zeros to be annexed as above, without actually inserting them. Should the division not terminate of itself, but admit of being carried on to any extent, then, at whatever point we stop the work, an unused *remainder* will be left; so that the decimal quotient will not, in strictness, be the *complete* value of the fraction; but the process may always be extended so far as to render the *correction* of the quotient too minute to be worth notice. It may indeed be made to become as small as we please. If the fraction be an improper fraction, the quotient will of course be partly integral and partly decimal, the decimal point occurring as soon as we begin to add zeros.

Ex. 1. Reduce $\frac{7}{8}$ to a decimal.

$$8\overline{)7000} \quad \therefore \frac{7}{8} = \cdot875.$$
$$\cdot875$$

2. The value of $\frac{64}{5}$ is 12·8.

3. In like manner, $\frac{15}{8} = 1{\cdot}875$, and so on.

4. Reduce $\frac{15}{256}$ to a decimal.

From the operation in the margin, it appears that 150 is too small to give a significant figure, so that the first figure of the quotient is 0. We see also that the remainders become exhausted only after eight decimal places are obtained in the quotient: the value of the proposed fraction, which might have been got by short division, is therefore, accurately, ·05859375.

```
256)1500(·05859375
    1280
    ----
     2200
     2048
     ----
      1520
      1280
      ----
       2400
       2304
       ----
         960
         768
        ----
        1920
        1792
        ----
         1280
         1280
         ----
```

(80.) The value of the last decimal is so small a part of unity, namely, the part $\frac{5}{100000000}$, as to be in most practical matters quite unworthy of consideration; we might therefore have stopped the process before arriving at this place, without troubling ourselves to see whether the work would spontaneously terminate or not. When an end is in this manner put to the operation, it is customary and proper to notice what the next figure *would be* if another step in it were to be made. Should this next figure prove to be a 5, or a figure still greater, then the figure at which we stop is increased by 1, because by so doing we secure the greatest possible accuracy for our result, as far as the operation has been carried. Thus, if we had stopped at the 7, in the present instance, we should have changed the 7 into 8, foreseeing, as we might, that the next figure would be 5, and knowing that if this 5 had been brought out, a 5 added to it would have converted the 7 into 8, while a 5 taken away from it,—that is, the suppression of the 5, would have left the 7 as it is. The error of adding a 5 or taking it away is, of course, the same; only in the one case it is an error in excess, and in the other case an error in defect. So far as accuracy, or the nearest approach to the truth, is concerned, it is *here* matter of indifference which plan we adopt; but it seldom happens that the work terminates at the figure next to that at which we stop; more figures would in general follow; so that the pro-

bability is, that by increasing our last figure by unit, when we foresee that the next figure that would arise is a 5, our error in excess is *less* than our error in defect would be if we were to suppress this 5, and all the following figures with it, and leave our last figure unaltered. When the next figure is greater than 5, the propriety of the alteration is obvious; so that by increasing the figure at which we stop by 1, whenever the next figure is foreseen to be 5 or greater, we *generally* attain a closer approach to the truth than we should do by leaving our last figure unaltered, and can *never* be farther from the truth. If the preceding result had been restricted to six places of decimals, it would have been ·058594; if it had been restricted to five places it would have been ·05859; if to four places, ·0586; and so on. And such is always the plan whenever superfluous decimals are suppressed. You will have to observe it in working the examples below.

(81.) The reason of the operation just performed is easily explained: the fraction $\frac{15}{256}$ is equal to $\frac{1500000000}{25600000000}$; that is, dividing numerator and denominator by 256, it is equal to $\frac{5859375}{100000000}$, which, in the decimal notation, is ·05859375; thus the equivalent decimal is obtained by dividing the 15, with the requisite number of zeros annexed, by the 256. And a similar explanation applies to every case.

Exercises.

Reduce to decimals the following fractions, namely,

1. $\frac{7}{16}$. 2. $\frac{5}{64}$. 3. $\frac{67}{125}$. 4. $\frac{93}{250}$.

5. $\frac{4}{625}$. 6. $\frac{14}{3125}$. 7. $\frac{10\frac{3}{4}}{20}$. 8. $\frac{4\frac{1}{5}}{758}$.

9. $\frac{3}{5}$ of $\frac{5}{7}$. 10. $\frac{2}{3}$ of $\frac{2\frac{1}{2}}{3\frac{1}{4}}$. 11. $\frac{4}{5}$ of $\frac{11}{12}$. 12. $\frac{3}{8}$ of $\frac{2}{5}$ of 6.

(82.) Addition and Subtraction of Decimals.

Very little need be said as to the addition and subtraction of numbers involving decimals. Just as in integers, we must be careful in these operations to place units under units, tens under tens, and so on; so here we must, in addition to this, be also careful to place tenths under tenths, hundredths under hundredths, and so on: that is, we must keep the decimal points all in the same vertical line or column, as in the examples following.

Addition.		*Subtraction.*	
23·462	·3258	683·2031	·826542
7·38	4·70234	479·8627	·71836538
26·151	15·1602	203·3404	·10817662
53·84	·0043		
76	37·10021	2·06853	1·4310063
2·3584	18	1·72946	1·3648163
189·1914	75·29285	·33907	·06619

Exercises.

1. Find the value of 27·62+·358+17·3+61+·007+173·1.
2. 5862·93+38·041+1·01+176·4+·0004+265·04.
3. 385·02+18·176−7·03−11·11−21·625+5·328+·061.
4. 1·0628+123·51−26·04+13−18·261+12·403−·082.
5. ·623+·0042+·79−·31−·002+·11+·08−·0003.
6. 246+187−5·613−19·148−7·03−104·6+·0018.

(83.) Multiplication.

Rule. Place the multiplier under the multiplicand, regardless of the decimal points, and proceed as you would with integers. Having thus got the product, mark off from it as many decimal places as there are decimal places in both the factors together, and the correct product will be obtained. If the product terminate in zeros, *these* need not be inserted, but they must be taken into account in pointing off the decimal places; and if there should happen to be fewer figures in the product than there are decimals in the factors, zeros must be *prefixed* to the product to make up the deficiency; and the decimal point is to be placed before them.

Ex. 1. Multiply 325·201 by 2·43.

Here the product, disregarding the decimal points, being found, as in the margin, *five* figures of it are to be pointed off as decimals, because the number of decimals in both factors amounts to *five*.

From looking at the foot-note in next page, you will see that the work may be abridged, by simply multiplying the first partial product by 8.

```
  325·201
     2·43
  -------
   975603
  1300804
  650402
---------
790·23843
```

2. Multiply 4132·65 by ·346.

Here, proceeding as in the margin, the terminating zero of the product is suppressed as useless; but as five decimals are to be pointed off, of which one has already been removed, we point off but four.

```
4132·65
   ·346
-------
2479590
1653060
1239795
---------
1429·8969
```

The work of the next example following may be shortened by multiplying the first partial product by 3, as explained in the foot-note below.

3. Multiply ·217 and ·0431 together.

Here the product consists of but five figures, while there are seven decimals in the factors; therefore two zeros must be prefixed, and the decimal point placed before them. To show the correctness of the operation, it will be sufficient to examine what is done in an example. Thus, the factors in Ex. 1 are $325\frac{201}{1000}$ and $2\frac{43}{100}$; that is, they are $\frac{325201}{1000}$ and $\frac{243}{100}$; the product of these is $\frac{325201 \times 243}{100000}$; the five zeros in the denominator implying that when the multiplication in the numerator is performed, *five* places must be pointed off for decimals. And it is plain, that in all cases by treating the factors in this way, the divisor will be 1, with as many zeros as there are decimal places in both factors.

```
  ·0431
   ·217
   ----
   3017
   431
  862
--------
·0093527
```

Exercises.

1. 32·605 × 6·417.
2. 183·52 × ·734.
3. 43·92 × 2600.
4. ·038 × ·072.
5. ·0037 × ·00021.
6. 2·46 × ·321 × ·07.
7. 1·73 × ·032 × ·0105.
8. 24000 × ·0016 × ·35.
9. 2·016 × 3·004 × ·0756.
10. 273·4 × ·036 × ·004.
11. 21000 × 1·02 × ·0268.
12. 1·4 × ·04 × ·4 × ·004.
13. 71·380164 × 2·7354.*
14. 138·6147 × 5·2575 × ·03.

(84.) CONTRACTED MULTIPLICATION.

I have now to introduce to your notice some considerations of especial importance in the multiplication of decimals, to

* It may not be amiss to point out to the learner here, that whenever a multiplier has figures side by side, which, when taken by themselves, form a number that is a multiple of some other figure, or of a number

which I must urge your particular attention, because although commonly adverted to in books on Arithmetic, they are usually presented in a way very much calculated to mislead. I have already told you, that decimals are frequently—far more frequently than otherwise—given in an incomplete form, because the complete form would often require an interminable extent of figures. I have told you (page 116), that it is customary, in such cases, to write down only a limited number of decimals, and to compensate for the figures omitted, by adding a unit to the final figure we retain, whenever the row of omitted figures commences with 5, or a figure greater than 5; but to make no compensation when this commencing figure is less than 5. In the comparatively few cases that *may* occur, in which the decimals we work with are really *complete*, we may multiply such decimals together, as above, and may be sure that our products are correct in every figure: but in the great majority of instances actually presented to us in practice, the decimals are curtailed, as just noticed; and, consequently, the last figure of every such decimal is not strictly correct: the error *may* be to the extent of half a unit in the place that last figure occupies.

The last figure of our multiplier being thus, in general, incorrect, it is plain, that the entire row of figures produced from it must be incorrect; or, at least, except under very favourable circumstances, that is, except when the error in our multiplier is very minute, the whole row, after one or two of the leading figures of it, must be incorrect.

In like manner, the final figure of our multiplicand being erroneous, in multiplying it by the successive figures of our multiplier, the products which arise from it must need correction; so that when all the partial multiplications are executed, and we proceed to add up, we must feel that, for

formed by any of the adjacent figures, as in the present example, the work may be abbreviated into a sort of short multiplication, thus:—

```
       71·380164
         2·7354

    214140492   = the product by 3, or 300.
  1927264428    = the foregoing prod. by 9, for the 27, or 27000.
     3854528856 = the last product by 2, for the 54.
  --------------
  195·2533006056 = the complete product.
```

The work of Examples 9 and 14 may be abridged in the same way.

several steps of this addition, we are really adding up wrong figures; and, consequently, that, as far as the influence of these reaches, our final product must be erroneous. I will give you an example. Suppose we have to multiply 27·14986 by 92·41035; and suppose, moreover, that these decimals had resulted from curtailing others of greater extent; as, for instance, 27·149855213, &c., and 92·41034604, &c.: we should obtain the product of our proposed factors, as in the margin: but if, instead of five places of decimals in our factors, we had taken six, we should have had

27·14986	
92·41035	
1	3574930
8	144958
271	4986
10859	944
54299	72
2443487	4
2508·928	065051

$$27{\cdot}149855 \times 92{\cdot}410346 = 2508{\cdot}927|49439983.$$

It is very plain, therefore, that the small error introduced into the fifth decimal of the factors, employed in the margin, has sufficient influence on the product, to render all the decimals of it, after the first three, widely erroneous. Even the third decimal differs by a unit from the more correct product above; but this latter is itself not strictly accurate, because advanced decimals have still been omitted in the factors: if the 213, &c., had been included in our multiplicand, all the decimals, beyond the vertical line, that is, beyond the 7, would have been affected; so that the 4, at present next to the 7, would have been a 5: we may conclude, therefore, that the product found in the margin is true, to the nearest unit, as far as *three* places of decimals; but that all the figures beyond these three should be expunged, as necessarily erroneous. In most books of Arithmetic, you are told, that these advanced decimals should be omitted, because they are *superfluous*, giving to the result a degree of minute accuracy not usually requisite in practical matters: but you see, from what is here shown, that they should be omitted, because no confidence can be placed in them, because, in fact, they are all wrong, and are no more worthy of being retained in our result than any row of figures written down at random in their place.

(85.) The practical conclusion you are to draw from what has now been said is this: when you multiply two factors together, the decimals in which have been curtailed, as here supposed, in adding up the partial products, disregard the

sums of all the columns up to that column, inclusive, in which the final figure of the *last* partial product is placed, and retain only the decimals furnished by the remaining columns. The last of the decimals thus retained should be increased by unit, if the first of the dismissed figures be a 5, or a greater number. In the marginal work above, a vertical line is drawn, cutting off the columns, of which the sums contribute nothing but inaccuracy to the result. It will, of course, occur to you, that it would save much waste labour if we could be spared the work of these inaccurate columns; and you will be glad to find that this may be done by a very simple contrivance. It was easy to foresee, before commencing the operation above, that *seven* decimals would have to be suppressed in the result; and, therefore, that *three* decimals only were to be retained: our object, then, would be to limit our operation to just so much work as would be necessary to furnish us with these three decimals; but as it is desirable that we should know what the fourth decimal would be, in order that the third may be as correct as we can make it; that is, in order that the third may be increased by a unit, should the fourth be a high figure, we ought to be able to get *four* decimals in our result, and then to limit it to three, which may be presumed to be correct in the last figure to the nearest unit: we have, therefore, to multiply 27·14986 by 92·41035, so as to give only *four* places of decimals in the product: this is done as follows: place the units figure of the multiplier under the *fourth* decimal of the multiplicand, and then write all the other figures of the multiplier, so as that the entire row may be *reversed:* then, in proceeding with this inverted multiplier, observe the following caution: reject all the figures of the multiplicand which lie to the *right* of the figure by which you multiply, carrying, however, from these rejected figures, whatever would have been carried if they had been retained; and write the *first figure* you get, in each partial product, in *the same vertical line*, as in the margin, and you will thus find the product, 2508·928, true to three places of decimals. The figure in the units place of the given multiplier being 2, this 2 is first put under the *fourth* decimal figure of the multiplicand; after which the inverted multiplier is completed, and the work

27·14986
5301429
24434874
542997
108599
2715
81
14
2508·9280

carried on agreeably to the preceding directions. By comparing it with the more lengthy operation before given, you will see that the partial products, as far as they are required, arise in reverse order, and are correct, as far as they go, to the nearest unit: you will, of course, observe, that, in the *carryings* from the rejected figures of the multiplicand, the uniform principle of compensating for a rejected 5, or greater figure, by adding a unit to the figure on the left of it, is to be attended to, and applied: thus, in multiplying by the 1, the product, arising from the 9 in the multiplicand, on the right, is rejected, but a compensating 1 is carried to the next product; that is, we say, once 4 is 4, and 1 carried makes 5. In like manner, when we reach the last figure, 5, we say, 5 times 7 are 35; carry 4: 5 times 2 are 10, and 4 are 14.

This example, with the explanations that have accompanied it, will sufficiently prepare you for the following rule.

(86.) *To find the Product of Two Factors, containing Decimals, to a proposed Number of Places.*

RULE 1. Count, from the decimal point in the multiplicand, as many decimals, annexing zeros if the decimals are too few, as you wish to secure decimal places in the product.

2. Under the last of these, put the *unit-figure of the multiplier*, or a zero, if there be no unit-figure, and then introduce all the other figures of it, so that the entire multiplier may appear with its figures *in reverse order*.

3. Multiply by the several figures of this reversed multiplier, neglecting, however, all those in the multiplicand to the *right* of the figure you are using, but, at the same time, carrying what would be carried, if nothing were neglected, and, moreover, carrying an additional unit, if 5 or a greater figure be rejected from the product.

4. Let each terminating figure of the partial products thus found, be in one vertical column; the first column to be summed up in the addition process: then, when this process is completed, you will have the product required, the decimal point being so placed, as to mark off the proposed number of decimals.

(87.) When the decimals in each of your factors are strictly true to the last figure, and your product is to be applied to a purpose, for which so many exact decimals as would make up the number in both the factors are not necessary, you may, by this rule, limit the number brought out to as few as you

please. It is, therefore, matter of choice with you, whether, in such a case as this, you take the trouble to work your example in full, and thus give to your result a needless degree of minute accuracy, or content yourself with only the amount of accuracy really wanted; and use, for this end, the contracted method: but remember, you have no choice when the decimals in your factors are not thus each of them complete and accurate in the final figure; you must then use the contracted method, not to dispense with needless accuracy, as above, but in order to preclude absolute error. In this case, you should count *all* the figures of that factor which contains the greater number, and so many figures of the uncontracted product, cut off from the right hand, should be expunged, not as merely *useless*, but as *erroneous*. You must, therefore, so apply the preceding rule, as to exclude, from the product, just this number of figures. NOTE, If one of the factors be quite correct, then only so many figures as *this* contains are to be rejected.

The following examples will sufficiently illustrate the application of the rule.

1. Multiply 348·8414 by 51·30742, so as to preserve only four decimals in the product.

$$\begin{array}{r} 348{\cdot}84140 \\ 2470315 \\ \hline 174420700 \\ 3488414 \\ 1046524 \\ 24419 \\ 1395 \\ 70 \\ \hline 17898{\cdot}1522 \\ \hline \end{array}$$

Here, reversing the multiplier, after having taken care to put the 1 in the multiplier, under the *fourth* decimal of the multiplicand, we see, that a vacant place occurs in the multiplicand over the final figure 5 of the reversed multiplier: we therefore supply this vacant place, by putting in it a 0, and then multiply, as in the margin. The result may be considered as correct, as far as it goes, provided the factors producing it, have no error in their last decimals; but, if we are not assured of this, then we cannot depend upon more than the first two decimals, for since the complete product would have *nine* decimals, and that each factor has *seven* figures, 9 — 7 = 2, expresses the greatest number of decimal places in the product that can be relied upon, with any confidence. The work would, therefore, be as here annexed: the 1 in the multiplier being now placed under the second decimal of the multiplicand. As the 2 in the multiplier has no figure above it, the product by this 2 is, of course, 0, but as the

$$\begin{array}{r} 348{\cdot}8414 \\ 24703150 \\ \hline 1744207 \\ 34884 \\ 10465 \\ 244 \\ 14 \\ 1 \\ \hline 17898{\cdot}15 \\ \hline \end{array}$$

product of the preceding 3 by the 2 is so great as 6, we carry 1 on that account, and the insertion of this 1 completes the multiplications.

2. Find the product of ·339377 and 325, to as many places of decimals as can be depended upon.

Here, as the multiplier, which is without error, has three figures, and as there are six decimals in the factors, only three decimals are to be preserved in the product, which is, therefore, 110·297. If we had computed to *four* places of decimals, we should have got 110·2975. As already noticed, we may always compute to one place more than the number of places to be preserved, and may increase the last of the preserved figures by unit, if the additional figure be so great as 5: in the present case, 110·297 and 110·298, may be considered to be about equally correct.

·339377
5230
101813
6787
1697
110·297

Exercises.

1. Multiply 480·14936 by 2·72416, and retain in the product only four decimals.
2. Multiply 15·917127 by 30·31667, retaining as many decimals as may safely be depended upon.
3. Multiply 1·7958563 by 30·31667, to four places of decimals.
4. Multiply ·62311052 by 170, to six places of decimals, which is one more than can be strictly depended upon.
5. Multiply 1·628894 by 214·87, retaining no decimals that cannot be relied on.
6. Multiply 81·4632 by 7·24651, retaining only three decimals.
7. Multiply 3·7719214 by ·4471618, retaining all the decimals to be depended upon, namely, six.
8. Multiply ·053407 by ·047126, retaining all the decimals that are likely to be correct.
9. Multiply 325·701428 by ·7218393, preserving only three decimals in the product.
10. Multiply ·63942, &c. by ·53217, &c.

NOTE. It is proper to state here, that from our ignorance of the true value of the decimals suppressed in our factors, and compensated for by a modification of the last decimal that is retained, and from the like modification of the last decimal in certain of our partial products, we cannot always be quite sure that the last decimal in our contracted product is invariably true to the nearest unit. It may in unfavourable cases err to the

extent of a unit; but it may generally be relied on as the true product to within this amount of error. An error to the extent of two units in the last figure is highly improbable. In thus speaking of the occasional departure from strict accuracy in the final decimal of our contracted product, it is to be understood, that the accuracy adverted to is that which the result would have if the suppressed decimals in the factors were restored. If it be of consequence, in any particular calculation in which we may be engaged, that the final decimal preserved in the product should be strictly correct, the safest way will be to compute, by the contracted method, to one or two decimals *beyond* those which are to be preserved, and then to dismiss them from the product. Many important money calculations are performed by decimals; and it is necessary that the computer should be cautioned against the very prevalent mistake of supposing that his accuracy is increased as he increases the number of the retained decimals. The contractions in this article are recommended, not on the score of brevity, but with a view to the securing of strict truthfulness as far as it is attainable. An error of a unit in the second decimal of a result expressing *pounds* is only about $2\frac{1}{4}d.$: a like error in the third decimal is less than one farthing. In any inquiry in which it is of consequence to secure accuracy in the decimals or integers of a product, beyond the places furnished by contracted multiplication, we may pronounce such accuracy to be unattainable, till our factors, erring as they do in the final decimal, be rendered more correct by the insertion of additional places. It is most remarkable, that in many of the modern books on arithmetic, in extensive use in instruction, not a word is said about contracted multiplication and division of decimals; the time and labour of the learner is occupied in working out long strings of figures, which the authors ought to know are all worthless, because all wrong; and what is worse, the pupil is all the while under the delusion that this useless labour is essential to the accuracy of his result.

(88.) Division of Decimals.

Rule 1. If the divisor and dividend have not the same number of decimals, annex zeros to make the number of decimal places equal.

2. Imagine the decimal points to be suppressed, and proceed as in division of integers, only with this difference, namely, if the divisor be greater than the dividend, annex a zero to the dividend: if the divisor be still the greater, put zero for the first figure of the quotient, and annex another to the dividend; and so on, putting a zero in the quotient for every zero annexed to the dividend, after the first.

3. Having thus made the dividend sufficiently great (disregarding the decimal points) to contain the divisor, carry on the work as with whole numbers, annexing a zero to every

remainder that arises after the figures of the dividend have been brought down, till as many decimals as are wanted are obtained in the quotient, or till the operation ends of itself. The number of zeros, employed in this latter way, together with whatever zeros the quotient may have commenced with, will be the same as the number of decimal places to be pointed off in the quotient. Or you may count all the decimals used in the dividend, including, of course, every zero annexed to a remainder: the difference between this number of decimals and the number of decimals in the given divisor will be the number of decimals in the quotient.

NOTE. You will often find this latter to be the most convenient way of finding the places to be pointed off in the quotient, as you may then dispense with adding zeros to the divisor when it has fewer decimals than the dividend.

Ex. 1. Divide 721·17562 by 2·257432.

```
2·257432)721·17562(319·467, &c.
         6772296
         -------
          4394602
          2257432
          --------
          21371700
          20316888
          --------
           10548120
            9029728
           --------
            15183920
            13544592
            --------
             16393280
             15802024
             --------
               591256
                 &c.
```

Here the operation in the margin has been carried on till nine decimal places of the dividend have been used, namely, the five decimal figures originally given, and four zeros besides. And since there are six decimal places in the divisor, three places must be pointed off in the quotient, agreeably to the note above. If we had proceeded strictly by the rule, and had written the dividend 721·175620, in order to make the number of decimal places the same as in the divisor, we might have introduced the decimal point in the quotient as soon as the dividend thus written had been exhausted; that is, as soon as the third figure 9 was found.

2. Divide 5·714 by 8275.

Here, it is easy to see that the quotient will commence with zero; therefore annexing three zero decimals to the divisor, since there are three decimal places in the dividend, and imagining the decimal points removed, we proceed by the rule, as in the margin, adding four zeros to the dividend, in order to make it large enough for the divisor, and consequently putting three zeros in the quotient. It is usual, however, to omit the zeros in divisor and dividend, and to proceed as if they were inserted; or, having no regard to the terminating zeros in the divisor at all, to perform the work agreeably to the note, leaving a gap at the beginning of the quotient, if a zero is foreseen to commence it, and then to complete the decimals as the note directs.

```
8275·000)5·7140000(·0006905
            49650
            -----
             74900
             74495
             -----
               42500
               41375
               -----
                1125
```

3. Divide ·079085 by ·83497.

The work of this example, freed from unnecessary zeros, is as in the margin: the last decimal 6 is a little too great, but is much nearer the truth than 5. The zero with which the quotient commences is put in last. We see that the eleven decimal places used in the dividend, diminished by the five in the divisor, leaves six for the number of places in the quotient, so that a zero must be prefixed to the decimal figures to make up the requisite number.

```
·83497) ·079085 (·094716
         751473
         ------
          39377
          333988
          ------
           59782
           584479
           ------
            13341
             83497
             -----
             49913
             500982
             ------
```

(89.) The reason of the directions given in the rule is obvious, when the decimal places in dividend and divisor are made the same in number, by the addition of zeros should need be; the suppression of the decimal point is merely equivalent to multiplying *each* by unity, followed by as many zeros as there are decimal places in each; so that the value of the quotient is undisturbed by these changes. It follows

also from multiplication, that the decimal places in both divisor and quotient must make up the number of places in the dividend.

The quotients obtained in the preceding examples are true in all the decimals only on the supposition that the final decimal in both dividend and divisor is strictly correct. This however is not generally the case; as it usually happens that a number consisting of several places of decimals is only an abridgment of a number with more decimals, as already explained. The last decimal of such a number is therefore always affected with some error,—some fractional part of the unit to which it belongs; and to prevent the influence of this error in the quotient of two such numbers, all that part of the work which the error affects should be lopped off, and the operation confined within trustworthy limits, as in contracted multiplication. In the following examples the decimals are supposed to be unaffected with error.

Exercises.

Find each quotient to three places of decimals.

1. 267·15975÷1·25.	4. 519·62÷7849.
2. 5·474558÷·0325.	5. 47·298÷6·029.
3. 325÷1·0125.	6. 3176·82÷·09317.

(90.) Contracted Division.

Contracted Division, like Contracted Multiplication, is a method of finding the result sought to as many figures as can be safely depended upon, without introducing into the operation any more work than what contributes to this object. The rule is as follows:—

Rule 1. Find the first figure of the quotient as in the uncontracted method, and thence the first remainder.

2. Instead of annexing a new figure from the dividend, or a zero, to this remainder, keep it as it is, and employ the divisor *with its final figure cut off* for the next step.

3. In like manner use the second remainder, and the divisor with two figures cut off for the next step; and so on till all the figures of the divisor have been dismissed. It must be observed, that what would be *carried* from the figure cut off step after step, is still to be carried in multiplying the quotient figure by the curtailed divisor.

If the divisor have more figures than the dividend, zeros may, of course, be annexed to the dividend; but then that portion of the work in which these zeros are brought into use cannot be depended upon, unless the last figure of the dividend be strictly accurate. When such is not the case, therefore, the overplus figures of the divisor should be cut off before commencing the division, if incorrect figures are to be excluded from the quotient.

1. As an example, let that at page 127 be taken; that is, let the value of 721·17562÷2·257432 be required to as many decimals as can be depended upon.

As no decimals are to be brought down to be annexed to the remainders, the final 2 in the dividend is suppressed as useless. The first figure 3 of the quotient is multiplied into the entire divisor as it stands; for the next quotient figure 1, the divisor is curtailed by a figure, and a dot is put under this figure 2 to remind us of this: for the quotient figure 9, the corresponding divisor is 2·2574, another dot being put under the 3, to imply that it is no longer to be used, except for what is carried from it: as $3\times 9=27$, the figure carried from it is 3. In this manner the operation is continued till the divisor is reduced to the single figure 2, and the work ends. This divisor 2, of the last dividend 6, gives only 2 for quotient; because if we try 3 we find that 1 must be carried from the divisor figure last rejected; so that the product would be 7. It is probable, however, that 3 is nearer the truth than 2; but the *last* figure of the quotient found in this way cannot, in general, be depended upon as strictly true. Suppose, for instance, that the final 2 in the divisor should be in strictness $2\frac{1}{3}$; then, what is now a 6 at the close of the work should be a 5, and the quotient figure 2 would be correct; but if the final 2 in the divisor should be $2-\frac{1}{3}$ or $\frac{5}{3}=1{\cdot}66$, &c., then the 6 at the

```
2·257432)721·1756(319·4672
  ······ 6772296
         -------
          439460
          225743
          ------
          213717
          203169
          ------
           10548
            9030
           -----
            1518
            1354
            ----
             164
             158
             ---
               6
```

end should be a 7, and 3 would be the correct quotient figure; so that here, as in contracted multiplication, the last decimal in the result may err by a unit.

2. Divide 7·66858 by $\frac{3\frac{1}{4}}{100}$.

Here, $3\frac{1}{4}=3·25$, and $\frac{3·25}{100}=·0325$; and as this decimal is strictly correct, the operation should proceed without contraction till the final figure 8 in the dividend has been used; after which, contraction must commence, as in the margin.

```
·0325)7·66858(235·956
   ··  650
       ----
       1168
        975
       ----
        1935
        1625
        ----
         3108
         2925
         ----
          183
          163
          ---
           20
           19
           --
            1
```

3. Divide 18·75, in which the 5 is not strictly correct, by 2·01747.

Here, instead of adding three zeros to the dividend, as we should do, or conceive to be done, if the last figure of the dividend were strictly accurate, we cut three figures from the divisor, and proceed as below, taking care, in multiplying by the first quotient figure 9, to carry what arises from the dismissed figures of the divisor, namely 7. If the final decimal 5 of the dividend had been quite accurate, the operation would then have been as here annexed, and the quotient may be considered as perfectly accurate, as far as four places of decimals; namely, 9·2938.

```
2·01,747)18·75(9·29        2·01747)18·75(9·29383
  ··       1816              ·····   1815723

           59                         59277
           40                         40349
           --                         -----
           19                         18928
           18                         18157
           --                         -----
            1                           771
                                        605
                                        ---
                                        166
                                        160
                                        ---
                                          6
```

From the illustrations now given, you can find no difficulty in multiplying and dividing decimals, which are not in themselves strictly correct in the final figures, so as to secure the greatest possible accuracy in your results. The subject is one of very great importance, and it therefore deserves your careful attention.

In the following exercises the final figure of each decimal is supposed to be more or less inaccurate, except when otherwise stated.

Exercises.

1. 31·782÷4·817.
2. 2490·3048÷·573286.
3. 2·149÷500·78.
4. 47·298÷6·029.
5. 4650·75÷325.
6. 8·6134÷7·3524.
7. 16·804379÷3·142.
8. 673·1489÷·41432.
9. 2·7182818÷3·1415927.
10. ·00128÷8·192.
11. 1708·4592÷·00024.
12. ·3412÷8·4736.
13. 75·347÷·3829.
14. 1÷10·473654.
15. 5·474558÷$3\frac{1}{4}$.
16. $\frac{1}{8}$÷·1045.
17. $23\frac{3}{16}$÷87·64378.
18. 14·3589÷·7854.
19. 2972160÷31773·244.
20. 103·936÷1059·108.
21. Perform Ex. 7 on the supposition that the final figure 2 of the divisor is strictly accurate.
22. Perform Ex. 20 on the supposition that the final figure 6 of the dividend is strictly accurate.
23. Perform Examples 3 and 12 on the supposition that each dividend is strictly accurate.
24. The old wine-gallon contained 231 cubic inches; the new or imperial gallon contains 277·274 cubic inches, the third decimal, however, 4, being a little too great: it is required to find how many imperial gallons are contained in the old wine-hogshead of 63 wine-gallons, old measure.

(91.) *Application of Decimals to Concrete Quantities.*

The application of Decimals to Concrete Quantities, is so like the application of whole numbers and common fractions, as to render any distinct rules here unnecessary: it will be sufficient to present to you a few examples, worked at length, as specimens of the operations.

Ex. 1. Find the value of ·761£.

The work is in the margin, and consists, as in common reduction, in simply reducing pounds to shillings, pence, and farthings. The answer is 15*s*. $2\frac{1}{2}$*d*., and the decimal, ·56 of a farthing, or 15*s*. $2\frac{3}{4}$*d*. nearly.

2. How much is ·37 of 15*s*. 5*d*.?

Here, 15*s*. 5*d*. = 185*d*., and 185*d*. × ·37 = 68·45*d*. = 5*s*. $8\frac{1}{4}$*d*., and ·8, that is, $\frac{4}{5}$ of a farthing, or 5*s*. $8\frac{1}{2}$*d*. nearly.

Ex. 1.	Ex. 2.
·761 £	185 *d*.
20	·37
15·220 *s*.	1295
12	555
2·64 *d*.	68·45 *d*.
4	4
2·56 *f*.	1·80 *f*.

3. What decimal of £3 7*s*. is £1 2*s*. 3*d*.?

Here, as in fractions, the two quantities must be reduced to a common denomination, and then the latter divided by the former: it is matter of choice, whether you bring them to the *lowest* denomination mentioned or to the *highest*: the work by both methods is given in the margin: in the first method, the quantities are reduced to *threepences*; in the second, to *pounds*: in the second, the 7*s*. is converted into ·35£; the 3*d*. into ·25*s*., to which the 2*s*. is prefixed; and then, the entire number of shillings, namely, 2·25*s*., brought into ·1125£; so that the proposed quantities, in the denomination, *pounds*, are 3·35£ and 1·1125£; the latter, divided by the former, gives ·3321, true to the nearest unit in the last decimal; the 1 is a little too great, but the error would have been greater if 0 had been put instead.

£.	*s*.		£.	*s*.	*d*.
3	7		1	2	3
20			20		
67			22		
4			4		
268			) 89(·3321		

804
86
804
56
536
24

2,0)7 *s*.
·35 £
12)3 *d*.
2,0)2·25 *s*.
·1125 £

3·35)1·1125(·3321
1005
1075
1005
70
670
30

4. What decimal of £5 is £3 17*s.* $6\frac{3}{4}$*d.*? Here, the shortest way is to proceed according to the second of the above methods, and to reduce the 17*s.* $6\frac{3}{4}$*d.* to the decimal of a £, as in the margin, and then to divide by the £5. The denomination, £, here placed against dividend and divisor, might have been omitted, since the quotient is the same abstract number whether dividend and divisor be concrete or not.

4)3
12)6·75
2,0)1,7·5625
£5)3·878125£
·775625

5. Reduce 2*s.* $9\frac{3}{4}$*d.* to the decimal of 7*s.* $9\frac{3}{4}$*d.*

Here, it would seem, that the best way is to reduce first to the lowest denomination, *farthings*, and then to divide the former quantity by the latter: it is plain, however, that the two may be a little simplified, by dividing each by 3: thus, $\frac{2s.\ 9\frac{3}{4}d.}{7s.\ 9\frac{3}{4}d.} = \frac{11\frac{1}{4}d.}{2s.\ 7\frac{1}{4}d.} = \frac{45f.}{125f.} = \frac{9}{25}$; we have, therefore, merely to turn the fraction, $\frac{9}{25}$, into a decimal, by actual division, as in the margin; so that 2*s.* $9\frac{3}{4}$*d.* is ·36, that is, 36 *hundredths* of 7*s.* $9\frac{3}{4}$*d.*

5)9
5)1·8
·36

6. Reduce 7 drams to the decimal of 1 lb. avoirdupois.

Here, $\frac{7}{16 \times 16} = {\cdot}027344$.

7. Reduce 14 min. to the decimal of a day.

Here, $\frac{14}{60 \times 24} = \frac{7}{720} = {\cdot}0097$.

8. Find the value of ·0125 lb. troy.

Here, $\cdot0125 \times 12 \times 20$ dwt. $= 3$ dwt.

9. If $\frac{3}{8}$ yard cost $\frac{7}{25}$£, what will $1\frac{2}{5}$ yard cost?

Here, $\frac{3}{8} = {\cdot}375$; $\frac{7}{25} = {\cdot}28$, and $1\frac{2}{5} = 1{\cdot}4$; $\therefore$ ·375 : 1·4 :: ·28£ : $\frac{\cdot28 \times 1\cdot4}{\cdot375}$ £ $=$ £1 0*s.* 11*d.* *nearly.*

Ex. 9.
·28 £
1·4
112
28
·375)·392(1£
375
17
20
·340(0 *s.*
12
4·08(11 *d.* nearly

Exercises.

Required the values of the following decimals, &c.

1. ·09375 acres.
2. 3·6285 degrees.
3. ·4625 tons.
4. ·4375 shillings.

5. ·73125 of £2 10*s*.
6. ·4694 lb. troy.
7. £19 17*s*. $3\frac{1}{4}$*d*. in *pounds*.
8. $\frac{14}{191} \times \frac{21}{758}$ in decimals.
9. Reduce 9*d*. to the decimal of a £.
10. Reduce 5 h. 48 min. 49·7 sec. to the decimal of a day.
11. What is ·315 of 2 lbs. 7 oz. 15 dwt. *troy?*
12. Multiply £3 4*s*. 6*d*. by 1·46875, and find the product in £ *s*. *d*.
13. Divide £10 11*s*. 3*d*. by 29·25, and find the quotient in £ *s*. *d*.
14. What common fractions are equal to 1·36 and ·1634?
15. The time between one new or full moon and the next is 29·5305887 days: reduce the decimal to hours, minutes, and seconds.
16. The *circumference* of a circle is 3·1416 times the *diameter:* the earth's circumference is about 24857 miles: find its diameter, as near as can be depended on; the 6, in the foregoing decimal, being slightly too great.
17. The diameter of the sun is about 883220 miles: find its circumference, as near as can be trusted (see Ex. 16).
18. What is the value of ·121875£+17*s*. $6\frac{3}{4}$*d*.?
19. What is the value of ·875£+·37 crown?
20. Work the following by decimals: If $2\frac{1}{2}$ qrs. of sugar cost £1 17*s*. 6*d*., what will 1 cwt. 3 qrs. 21 lb. cost?
21. If 24 ac. 3 ro. 39 per. can be reaped in $12\frac{1}{2}$ hours, how much can be reaped by the same hands in 15 h. 48 min.?
22. If £6 13*s*. be the wages of 8 men for 3·25 days, what will be the wages of 20 men for 9·25 days?
23. Find the value of ·34 of ·26 of £2 13*s*. 1*d*.
24. Reduce ·47 of ·23 of 7*s*. $1\frac{1}{2}$*d*. to the decimal of £1 14*s*. $8\frac{1}{2}$*d*.

(92.) *Recurring, or Circulating Decimals.*

Before concluding the arithmetic of decimals, it is proper to say a few words about what are called *recurring*, or *circulating* decimals. They are so called, because the figures of which they consist continually recur, presenting either a constant repetition of the same figure, after a certain number of figures, or a repetition of the same set or row of figures: thus, ·3333.... is a recurring decimal; so also is ·7543543..., and ·592592...., &c. In the first of these

instances, 3 is the recurring figure; in the second, 543 is the recurring *period*, as it is called; and in the third, the recurring, or circulating period, is 592. Instead of repeating the period, it is customary to write it but once, and to put dots over the extreme figures of the period, by which we are to understand, that those figures recur without end: the three instances just noticed would thus be more briefly expressed as follows: $\cdot\dot{3}$, $\cdot 7\dot{5}4\dot{3}$, and $\cdot\dot{5}9\dot{2}$. Decimals, of which the figures have this periodic character, very frequently present themselves in converting a common fraction into a decimal; indeed, they *always* present themselves whenever the denominator of a vulgar fraction, in its lowest terms, is not entirely resolvable into factors consisting of 2's and 5's. You are aware, that, in order to convert a fraction into a decimal, we divide the numerator by the denominator, continually annexing zero after zero to the former, till the operation terminates of itself, or till we arbitrarily put a stop to it: and it is plain, that if we are at liberty to put as many 0's as we please to a number, that number, with the zeros attached to it, will become divisible, without remainder, by as many 2's and 5's as we please. Hence, a fraction whose denominator has no other simple factors but these, is always equal to a *finite* or *terminable* decimal: but, if other factors enter, or, which is the same thing, if, after removing all the 2's and 5's, a factor still exists in the denominator, then, the decimal, equal to the fraction, will be *interminable*, because this remaining factor, not being divisible by either 2 or 5, must terminate, either in a 1, a 3, a 7, or a 9; and no quotient-figure, multiplied into either of these, can ever produce a 0, as the terminating figure of the product, so that we might bring down 0's continually, without any hope of the work ending of itself: the following are a few instances.

$\frac{1}{9} = \cdot 1111 \ldots = \cdot\dot{1}$	$\frac{83}{9768} = \cdot 008\dot{4}9713\dot{3}$
$\frac{1}{11} = \cdot 0909 \ldots = \cdot\dot{0}\dot{9}$	$\frac{7916}{1665} = 4\cdot 7\dot{5}4\dot{3}$
$\frac{16}{27} = \cdot 592592 \ldots = \cdot\dot{5}9\dot{2}$	$\frac{1}{19} = \cdot\dot{0}5263157894736842\dot{1}.$

As the last of these examples shows, a very simple fraction may give us a good deal of labour, before we can determine the circulating period of its equivalent decimal:* but, in a

* A method of abridging this labour was given by Mr. Colson, in Sir Isaac Newton's "Fluxions." An analogous method, much more convenient and expeditious, was proposed by the author of this Rudimentary Treatise, in Vol. xxxvi. of the Philosophical Magazine; the numbers for January and February, 1850.

case like this, such determination would be more curious than useful: it is easy to prove, however, that, the fraction being in its lowest terms, the period can never have so many figures as there are units in the denominator of the fraction: in the case of $\frac{1}{19}$, for instance, we might be sure that the period could not have more than eighteen figures, which number we see it actually contains. The reason is this: the period can extend itself only so long as the successive *remainders* we arrive at, in carrying on the division, do not *recur:* whenever we come to a remainder, the same as one already employed, then, of course, the quotient-figures between the two must also recur; but this recurrence is postponed so long as the remainders continue to be all *different;* and as no remainder can be greater than the number which is a unit less than the divisor, it is plainly impossible that there can be *more* different remainders than is expressed by the divisor, *minus* 1. And this is, in general, all that we can say about the extent of the period, previously to actual trial.

Although recurring decimals are thus always interminable, you are not to infer, that interminable decimals are always recurring; those only are recurring which arise from the *development*, as it is called, of a vulgar fraction; such decimals are always convertible back again into the finite fractions to which they are equivalent; but many interminable decimal values occur in calculation which cannot be represented by a finite fraction, and which, therefore, are not recurring decimals. Of such decimals, only a finite portion of the interminable row can be used in computation, so that some amount of error in the abridged forms is unavoidable: in the preceding articles, I have shown how to exclude from any result that part of it which this imperfection would influence. When the decimals with which we work are recurring, the imperfection, consequent upon our using only a finite number of figures, can be rendered as minute as we please; for one period being given us, we can always add on as many true decimals as we like, instead of employing zeros when we want additional places; to use zeros for such a purpose in *recurring* decimals, you will, of course, see, would be an intentional departure from strict accuracy: but, in dealing with circulating decimals, the way to avoid imperfection altogether, is to convert them into their equivalent vulgar fractions: the rule for this is as follows.

(93.) *To convert a Recurring Decimal into its equivalent Vulgar Fraction.*

RULE 1. Write down only the figures after the decimal point, up to the end of the first period, omitting the leading 0's, if there be any, and consider the number thus written to be a *whole number*.

2. Subtract from this whatever portion of the decimal there may be which *precedes* the period, regarding this portion as a whole number also; the remainder will be the *numerator* of the equivalent fraction.

3. For the *denominator*, write as many 9's as there are figures or places in the period, followed by as many 0's as there are decimals preceding the period. Prefix to the fraction whatever whole number may have been prefixed to the decimal.

Ex. 1. Convert $\cdot\dot{0}\dot{9}$ into its equivalent fraction.

Here, omitting the leading 0, we write 9 as a whole number; and as no decimals precede the period, the 9 will be the numerator of the required fraction, and 99 will be the denominator: therefore the fraction is $\frac{9}{99} = \frac{1}{11}$.

2. Convert $\cdot\dot{5}9\dot{2}$ into its equivalent fraction.

Here, agreeably to the rule, the fraction will be $\frac{592}{999}$, which, by dividing numerator and denominator by 37, reduces to $\frac{16}{27}$.

3. Convert $4\cdot7\dot{5}4\dot{3}$ into its equivalent fraction.

Here, the figures first to be written down are 7543, as a whole number, and from this, the whole number 7 is to be subtracted: the numerator is therefore 7536, and the denominator 9990; consequently, the fraction, with the whole number 4 prefixed, is $4\frac{7536}{9990} = 4\frac{1256}{1665} = \frac{7916}{1665}$.

Exercises.

Reduce to fractions the following decimals.

$\cdot\dot{1}3\dot{5}$, $2\cdot4\dot{1}\dot{8}$, $\cdot5\dot{9}2\dot{5}$, $\cdot004\dot{4}\dot{9}$, $3\cdot\dot{7}56\dot{9}$, $621\cdot\dot{6}2\dot{1}$, $\cdot\dot{0}243\dot{9}$, $\cdot\dot{8}5714\dot{2}$, $1\cdot\dot{0}37\dot{8}$, $\cdot008\dot{4}9713\dot{3}$.

To understand the principle of the rule, it will be sufficient to attend to the following simple and obvious cases, namely, $\frac{1}{9} = \cdot11111\ldots.$, $\frac{1}{99} = \cdot010101\ldots.$, $\frac{1}{999} = \cdot001001\ldots..$, $\frac{1}{9999} = \cdot00010001\ldots$, &c., from which it appears, that a recurring decimal, whose period commences immediately after the decimal point, is converted into an equivalent fraction, thus: if the period consist of but one figure, this figure, taken

as a whole number, must be multiplied by $\frac{1}{9}$; if it consist of two figures, these, taken as a whole number, must be multiplied by $\frac{1}{99}$; if it consist of three figures, the number must be multiplied by $\frac{1}{999}$; and so on: thus, $\cdot\dot{3} = \frac{3}{9}$ or $\frac{1}{3}$; $\cdot\dot{7} = \frac{7}{9}$; $\cdot\dot{5}9\dot{2} = \frac{592}{999}$, &c.; which is according to the rule. If the period do not commence immediately after the decimal point, as, for instance, in $\cdot 7\dot{5}4\dot{3}$, then $\cdot 7\dot{5}4\dot{3} = \frac{7\cdot\dot{5}4\dot{3}}{10} = \frac{7+\frac{543}{999}}{10} = \frac{7\times 999+543}{9990} = \frac{7000-7+543}{9990} = \frac{7543-7}{9990}$; which agrees with the rule. In like manner, $\cdot 27\dot{5}4\dot{3} = \frac{27\cdot\dot{5}4\dot{3}}{100} = \frac{27+\frac{543}{999}}{100} = \frac{27\times 999+543}{99900} = \frac{27000-27+543}{99900} = \frac{27543-27}{99900}$; and so on, agreeably to the rule.

(94.) INVOLUTION AND EVOLUTION.

WHEN a set of *equal* factors are multiplied together, this particular case of multiplication is called *involution*, and the product is called a *power* of the number or factor, thus repeatedly used. If the number be simply multiplied by itself, the product is the *second power*, or *square* of that number: if the second power be also multiplied by the number, the product is the *third power*, or *cube* of that number: if, again, *this* be multiplied by the number, the product is the *fourth* power of that number: and so on, the number of times we thus use the same number as factor, always marking the *power* of that number, so that numbers may be *raised* to the fifth, sixth, seventh, or any other power, however great, provided only we have the patience to carry on these successive multiplications by it.

As this operation of involution is no more than common multiplication, there is nothing for me to explain in reference to the mode of performing it; and I here mention it, chiefly for the sake of showing you the meaning of a term in frequent use, and of introducing another particular in notation.

From what has just been said, you see that the *second power*, or *square* of any number, 3, for instance, is thus indicated: $3\times 3 = 9$; that is, the square of 3 is 9: that the

third power, or cube of 3, is $3\times3\times3=27$; that is, the cube of 3 is 27: that the *fourth power* of 3 is $3\times3\times3\times3=81$; that the fifth power is $3\times3\times3\times3\times3=243$; and so on, to any extent. You see that it would soon become inconvenient and tedious to repeat the equal factors in this way, when high powers are to be indicated; and to avoid this, a very neat and brief form of notation for powers has been devised: the number to be *involved* or raised to the proposed power, is simply written down; and then, over the right-hand upper corner of it, is placed, in smaller type, another number, expressing the *power* intended: thus, 3^2 indicates the *square*, or *second* power of 3; 3^3 indicates the *cube*, or *third* power of 3; 3^4 indicates the *fourth* power of 3; 3^8 the eighth power of 3; 3^{12} the twelfth power; and so on. The number which, involved in this way, produces a power, is called the *root* of that power: thus, 3 is the *square-root* of 9; it is the *cube-root* of 27; the *fourth root* of 81; and so on: the notation for a root is the symbol $\sqrt{}$, which, when no small figure is attached to it, implies, simply, the *square-root;* when the *cube*-root is intended, a little 3 is connected with the symbol, thus $\sqrt[3]{}$; when the fourth root is meant, a little 4 is used, thus $\sqrt[4]{}$; and so on. You will now have no difficulty in making out the meaning of the following statements.

Since $3^2=9\ \therefore\ \sqrt{9}=3$; since $3^3=27\ \therefore\ \sqrt[3]{27}=3$; since $3^4=81\ \therefore\ \sqrt[4]{81}=3$; &c.

Since $5^2=25\ \therefore\ \sqrt{25}=5$; since $4^3=64\ \therefore\ \sqrt[3]{64}=4$; since $7^4=2401\ \therefore\ \sqrt[4]{2401}=7$; &c.

(95.) There is one thing that must occur to you in looking over these particulars: it is this,—that although the *power* proposed is very easily got from knowing the *root*, yet that it is not so easy to discover the *root* when we know only the *power;* thus, you would find it no easy matter to get the fourth root of 2401, had you not previously seen that it was produced by successive multiplications of 7 by itself. This *reverse* operation, by which any proposed *root* of a number is found, is called *Evolution.* The process of *Involution* is uniform, whatever be the power to which a number is to be raised; but it is not so with *Evolution:* the rule for the *square-root* would help you but little towards finding the *cube-root*, the *fifth root*, &c. I am now going to show you how the square-root of a number is to be found, and afterwards how the cube-root is to be found; but I must previously tell you that comparatively few numbers are really

squares or *cubes;* that is, numbers actually produced by *squaring* or *cubing* other numbers. We cannot, of course, find by rules what does not strictly exist; but by aid of decimals we can obtain *approximate* square-roots, and cube-roots of all numbers: that is, by applying our general rules we can, by means of decimals, obtain a number which, when *squared* or raised to the second power, shall produce a number differing from the number whose square-root is required by a fraction or decimal as small as we please. And we can also obtain a number which, when cubed or raised to the third power, shall produce a number differing from the number whose cube-root is required by a decimal as small as we please; so that *such* square and cube roots may be taken as *true* square and cube roots, without any sensible error. Decimals are very useful in all calculations, where *approximate* values only are attainable. The following is the rule for finding the square-root accurately of a number, whenever that number is strictly a square, and for finding the square-root *approximately* to any extent of decimals when the number is not an exact square.

(96.) *To extract the Square-root of a Number.*

RULE 1. Prepare the number for the operation thus:—Commencing from the decimal point, mark off the two final figures of the integral portion of the number; then the two figures which precede them; then the two before these; and so on, cutting up the integers in this way into as many periods, of two figures each, as you can: and, returning to the decimal point, mark off pairs of decimals, proceeding from left to right, in the same way. As we are at liberty to put a 0 at the end of the given decimals, we may always make the number of *them* even; so that the *decimals* will consist of complete periods, without any odd figure over; but if the integers be odd in number, then, besides the periods of two figures each, there will be the leading figure standing singly; this leading figure, however, is still called the *first period.*

2. Attend only to the first period, and find the *greatest number* whose square does not exceed the number in that period. This can never be matter of the slightest difficulty; for as the period can never be a number of more than *two* figures, it will be very easy to see which of the nine digits multiplied by itself approaches nearest to it. The greatets

number thus found is the first figure of the root: write it in the place in which you would put it if it were the first figure of a quotient; that is, to the right of the proposed number; subtract the square of it from the first period, and to the remainder annex *the second period*, and you will have a number which may be called *the first dividend.*

3. After this leading step the operation assumes a new form. To the left of the first dividend mark off a place for the corresponding first divisor, which you find thus:—Put twice the root-figure just found in the divisors place; the leading figure, or if the root-figure exceed 4, the *two* leading figures of the required divisor will thus be found, and you will now have a dividend, and the leading figure or figures of its divisor, to find the corresponding quotient-figure; and you know, from common division, that it is mainly the *leading* figures of a divisor which suggest the first figure of the quotient. Find then the quotient-figure from this *incomplete divisor;* the quotient-figure thus found forms the second figure of the root, and, annexed to the incomplete or *trial* divisor, it renders it *complete;* you have only then, as in division, to multiply the complete divisor by the figure just found, to subtract the product from the dividend, and to annex to the remainder *the third period;* you will thus have *the second dividend.*

4. Proceed step after step in this way, writing against every dividend twice the number formed by the root-figures previously found; you will thus always get the incomplete or *trial divisor* belonging to that dividend, and thence a new root-figure; with which, as before, the incomplete divisor is to be completed.

You know that in common division the quotient-figure suggested by the *leading* figures of a divisor is not always the *true* quotient-figure; for we cannot always foresee the full influence of the *carryings.* So here, the root-figure, suggested by an incomplete divisor, may prove to be erroneous when that divisor is completed, and the multiplication of it by the figure under trial executed. In such a case we do exactly as we would in common division. (See page 30.)

An example worked at length will sufficiently illustrate the rule.

Ex. 1. Extract the square-root of 56782·432.

```
          5ˎ67ˎ82ˎ·43ˎ20(238·2906
          4
        43)167
           129
       468) 3882
            3744
      4762)13843
            9524
     47649)431920
           428841
   4765806)30790000
           28594836
            2195164, &c.
```

Here, the proposed number divided into periods of two figures each, as the first precept of the rule directs, is 5ˎ67ˎ82ˎ·43ˎ20, and the first root-figure is 2, this being the greatest number, whose square (4) does not exceed the first period, 5: the square of this 2, subtracted from the first period, leaves for remainder 1, which becomes 167, when the next period is brought down. Hence, 167 is the first dividend, and 4, the double of the root-figure, is the first trial, or incomplete divisor. Looking at this 4, in reference to the 16 in the dividend, 4 is suggested as the quotient-figure; but, foreseeing that unit would have to be carried, we know that 4 will be too great. Putting, therefore, 3 for the second root-figure, and the same 3 against the incomplete divisor, we proceed, as in division, and obtain the second remainder, 38, which, when the next period is brought down, becomes 3882, the second dividend. Doubling the 23, the number formed by the root-figures already found, we have 46 for the incomplete divisor of 3882; so that the corresponding quotient-figure—that is, the third root-figure—is 8, which placed against the 46, gives 468 for the true divisor: hence, the third remainder is 138, and another period being brought down, the third dividend is 13843. The trial-divisor of this, that is, the double of the root, thus far found, is 476; and, therefore, the quotient-figure, that is, the fourth root-figure, is 2, and, therefore, the complete divisor is 4762. The next dividend is 431920, and the trial-divisor belonging to it is 4764, so that 9 is the fifth figure of the root: completing the divisor with this 9, we get the next remainder, 3079, which, with another period brought down, namely, 00, gives the next dividend, 307900, the incomplete divisor of which, the double of 23829, is 47658. It is

obvious, from the leading figure, that this, when completed, will be *greater* than the dividend: hence, the next root-figure is 0; and the next dividend, formed by annexing another period of zeros, is 30790000, the corresponding incomplete divisor being 476580, the double of the root, so far as it goes. The incomplete divisor gives 6 for the next root-figure, and 4765806 for the complete divisor; and, by bringing down zero periods in this way, step after step, we may extend the root to as many decimal places as we please. In the work in the margin, it has been carried to four places of decimals, and the number 238·2906, thus determined, is said to be the approximate square-root of 56782·432; and the decimals are all true, as far as these four places. Nevertheless, as more decimals would follow if we were to continue the work, the final figure, 6, needs a fractional correction; we know that it is too small, by some fraction, or decimal of a unit, in the fourth place. On this account, we could not expect to recover the proposed number exactly, by squaring this incomplete root.

From the principles already taught (page 120), you know that if you were to multiply 238·2906 by itself, the result could not be depended upon beyond the first decimal; for, as there are seven figures in each factor, and but eight decimals in the complete product, the number of decimals to be depended upon is only $8-7=1$, the other seven decimals being necessarily inaccurate. You must always keep in remembrance the influence of this error, in the last decimal of an approximate result, whenever you have occasion to use it in multiplication or division; and be careful to avoid the common mistake of supposing that, because you have got the square-root of a number true to several places of decimals, that the *square* of that root can be true to anything like the same extent. The square of the root just obtained, as far as one decimal place, which is all that can be safely depended on, is found by contracted multiplication, as in the margin.* If you take the trouble to multiply the root by itself, without any contraction, you will find the product to be 56782·41004836. By extending the decimals of the

```
238·2906
 6092832
--------
 4765812
  713872
  190632
    4766
    2144
      14
--------
55782·4
```

* See note, page 125.

root, the square of the root may be made to differ from the number proposed by a quantity smaller than any that can be assigned; so that you always have it in your power to *approximate* to the truth, as closely as you please.

(97.) We must now return to our worked-out example at page 143, for I have some particulars to mention to you, as to the abridgment of the operation. You may, in general, take it for granted, unless the contrary plainly appear, that when a number containing decimals is given you to work upon, the final decimal of that number will be only *approximately* true. You may consider, therefore, that the number proposed in the example referred to, has, in its complete state, a long string of decimals, and that the decimals have been reduced to three, for convenience, or, because more than three decimals would encumber the value with an unnecessary degree of accuracy; consequently, when we have used the final decimal 2, we may conclude, that the final figures of the subsequent dividends will be affected with error, and our business is, to exclude the influence of this error from the figures of the root. It is plain, therefore, that after having arrived at the dividend, 43192, and its divisor, 47649, *contracted division* only should be used: you must see, too, that as each divisor differs from the next *trial*-divisor following, only by having the double of the new root-figure *added* to it, these successive additions can have no influence on any *contracted* divisor, because the figures that *would be* influenced become cut off. Hence, after the final decimal 2 has been used, the remaining part of the operation is simply that of contracted division, which terminates of itself, as soon as all the root-figures that can be depended upon are obtained: the root may, therefore, be extended, as in the margin; and we may consider it to be correctly determined, as far as five places of decimals; its value being 238·29064: but as the quotient-figure 4 is somewhat more *below* the truth than 5 is *above* it, it might be better to write 238·29065 for the root; if we knew that the final decimal in our proposed number were a little *too small*, we should affirm the latter to be the better approximation to the root; but, if we knew that

```
4,7,6,4,9)43192(9064
          42884
          -----
            308
            286
            ---
             22
             19
             --
              3
```

our proposed number were a little too great in its last decimal, we should prefer the former approximation: we cannot, therefore, be quite sure as to the *last* decimal, within a unit.

In the following example, contraction is used as soon as the decimals in the given number have been employed.

2. Extract the square-root of 58352·74.

```
       5,83,52,·74(241·5631
       4
       —
     44)183
        176
        ———
    481)  752
          481

   4825)27174
         24125
         —————
 4,8,3,0) 3049
          2898
          ————
           151
           145
           ———
             6
             5
             —
             1
```

The work in the margin shows the root as far as four places of decimals to be 241·5631: you cannot depend upon it to any greater extent.

(98.) I think you must now sufficiently see the manner in which you are to proceed with the extraction of the square-root, when you desire that root to be encumbered with no more decimals than you can place reliance upon. You can, of course, adopt the same method of contraction, when you have to approximate to the square-root of a whole number, in itself not a square: as the process, left unchecked, would go on without end, you would fix a point at which restraint should be put upon the increase of figures; and, by using contracted division from that step, bring the operation to a close: you can always easily see, from counting the number of figures in any divisor, how many root-figures, from that stage of the work, will be added on by the contracted division.

After the first step in the extraction of the square-root, you need not take the trouble to multiply the root-figures by 2, in order to get the several trial, or incomplete divisors, since each trial-divisor is formed from merely adding twice the last root-figure to the preceding divisor; in practice, the several trial-divisors are always derived from one another in this way. I may also remark here, that some people mark off the periods, in the number proposed for extraction, by

putting a dot over the last figure of each period: thus, the periods of the number in last example are distinguished in this way; $\dot{5}8\dot{3}5\dot{2}\cdot7\dot{4}$: you can, of course, use, in your own practice, whichever way you please.

(99.) It is not easy to give a satisfactory proof of the foregoing method of extracting the square-root without the aid of algebra; but I will endeavour to explain the reason of the several steps of the operation by help of arithmetic only. To understand this explanation, you must first become convinced of the following property: namely, that if any number be separated into two parts by the sign *plus*, the square of that number will always be made up of the squares of the two parts *plus* twice the product of those parts. Take, for instance, the number 9, of which the square is 81; this square is made up of the squares of 4 and 5 (which together make 9) *plus* twice 4×5; it is also made up of the squares of the two parts 3 and 6 *plus* twice 3×6; or of the squares of the two parts 7 and 2 *plus* twice 7×2, and so on: that is, $9^2 = 4^2 + 2\,(4 \times 5) + 5^2 = 3^2 + 2\,(3 \times 6) + 6^2 = 7^2 + 2\,(7 \times 2) + 2^2 = 8^2 + 2\,(8 \times 1) + 1^2$, &c. And the same property holds, whatever be the number, and into whatever two parts it be separated. It is this general property that has suggested the rule for the square-root. Let us take any square number; the square of 8764, for instance; which is 76807696. If from having this square given, we wished to return to the root, we should readily foresee, that by dividing it into periods, as already explained, and then regarding only the first period 76, the leading figure 8 of the root could be at once discovered. As the local value of this 8 is 8000, we know, from the foregoing property, that the proposed number is made up of the following parts: namely, $8000^2 + 2\,(8000 \times 764) + 764^2$, which is the same as $8000^2 + (2 \times 8000 + 764)\,764$. So that after having subtracted the square of the first root-figure, the remainder, or what has been called above *the first dividend*, is $(2 \times 8000 + 764)\,764$.* The divisor for this, which would supply accurately all the remaining figures of the root, namely the figures 764, is evidently $2 \times 8000 + 764$. But this divisor we cannot completely get, as the figures 764 which enter it are those of which we are in search; but we can get the greater part of it; namely, the part 2×8000, from knowing the already-found first figure 8, or in strictness 8000. We avail ourselves, therefore, of this, and call it our first *trial*, or incomplete divisor; this assists us in discovering the single figure 7, by which we are enabled to correct our trial divisor, by adding to it the part thus suggested, namely 700; and so corrected, we call $2 \times 8000 + 700$ our *complete divisor*, since it completely answers for

$$
\begin{array}{r|l}
 & 76,80,76,96\,(8764 \\
 & 64 \\
\hline
167) & 1280 \\
 & 1169 \\
\hline
1746) & 11176 \\
 & 10476 \\
\hline
17524) & 70096 \\
 & 70096
\end{array}
$$

* The remainders, or dividends, are conceived to have the figures of the proposed number, afterwards brought down two at a time, to be actually appended to them; but, just as in long division, the remainders are kept free of these additional figures till they are wanted for use, in order to save unnecessary repetitions. It may be noticed here, that the notation above means that the whole of the quantity enclosed in the brackets is to be *multiplied* by 764.

the figure 7 of the root, though not for the figures which follow. After multiplying this complete divisor by the local value, 700, of the quotient-figure 7, we get $(2 \times 8000 + 700)\ 700$, which subtracted from the first dividend, leaves $2 \times 8000 \times 64 + 764^2 - 700^2$. But by the general property with which we started, $764^2 = 700^2 + 2\ (700 \times 64) + 64^2$: consequently, the remainder spoken of is $2 \times 8000 \times 64 + 2 \times 700 \times 64 + 64^2$; or, which is the same thing, $2 \times 8700 \times 64 + 64^2 = (2 \times 8700 + 64)\ 64$. And this is our *second dividend*. Now just return to the preceding expression for the *first dividend*, and you will observe a perfect correspondence; the first dividend consists of twice the number already in the root + the number formed by the remaining figures, *multiplied* by that number; so here the *second dividend* consists of twice the number already in the root + the number formed by the remaining figures, *multiplied* by that number. Consequently, just as we got the second figure of the root out of the first dividend, so we may now get the third figure out of the second dividend; that is, the step by which the third figure is to be obtained must be exactly similar to that by which the second was obtained; and therefore, figure after figure is to be found by a series of uniform steps, as in the operation given at length in the margin, which you will find upon examination to be in strict accordance with what is here explained, when the several remainders or dividends are completed, by the latter figures of the original number being appended to them. In the work, these figures, to save repetitions, are not actually brought down till they are wanted.

Exercises.

Extract the square-root of each of the following numbers, those of them which terminate in decimals being only approximately true in the last figure, except where otherwise stated.

1. 31·782153.
2. 115·297356.
3. ·3236068.
4. 11, to six decimals.
5. 473256, to three decimals.
6. 3, to eight decimals.
7. 903687890625.
8. 365, to eight decimals.
9. 32·398864, the last decimal *true*.
10. ·000729, the last decimal *true*.
11. 784·375
12. 79·182.
13. 68·736, to nine decimals, the last, 6, being *true*.
14. $29\frac{52}{81}$.
15. $104\frac{1}{25}$.
16. $17\frac{3}{8}$, to five decimals.
17. $15\frac{5}{8}$, to six decimals.
18. $794\frac{1}{5}$, to eight decimals.
19. 34·867844.
20. The recurring decimal $7\cdot6\dot{5}3\dot{1}$ to seven places.

(100.) There is another way of arranging the work for the square-root, which, although it presents to the eye more figures, and takes up more room, is nevertheless well worthy of your attention, on account of the very simple character of the several steps; and because, moreover, the operation, when

thus arranged, becomes only a particular case of the general process for the extraction of any root, however high. The operation, too, is the same, whether the sought root be that of a *number* merely, as here, or that of a *numerical equation;* so that, if you ever advance to algebra, you will find a familiarity with the arrangement given below to be of much assistance to you in an important part of that subject. The work of the example last given, in the form here recommended to your notice, is as follows:—

1	0	76,80,76,96(8764
	8	64
	——	——
	8	1280 (1)
	8	1169
	—— (1)	——
	16	11176 (2)
	7	10476
	——	——
	167	70096 (3)
	7	70096
	—— (2)	——
	174	
	6	
	——	
	1746	
	6	
	—— (3)	
	1752	
	4	
	——	
	17524	

By comparing this with the work of the same example at page 147, you will see that the two operations differ only in arrangement. In the latter form, 1, 0, and the given number, are written in a row; the 0 stands at the head of a column of work which gives the trial and true divisors; and the given number stands at the head of a column which furnishes the corresponding dividends; each trial-divisor, with the dividend to which it belongs, is marked (1), (2), &c, merely for the purpose of directing the eye to those numbers in the two columns which are directly concerned in the determination of the several root-figures; each true divisor is

always one step below the trial-divisor, which suggests it. The several steps of the work are of the simplest kind: having found the first figure 8 of the root, the 1 is multiplied by it, the product added to the 0, the result multiplied by the same 8, and the product *subtracted* from the first period: the 1 is again multiplied by the 8, the product, as before, carried to the divisor-column, and thus one step of the work is completed. The trial-divisor, 16, now suggests the next root-figure, 7, which, as before, is a multiplier of the 1; the product carried to the divisor-column gives the true divisor, and the product of this by the 7, carried to the dividend-column, and *subtracted*, gives the next dividend (2); the 1 is again multiplied by the 7, and the product carried to the divisor-column, completes the second step; and so on, all the steps being uniformly the same.

(101.) The operation for the cube-root of a number is merely an extension of this easy kind of work. In this case, the row at first written consists of 1, 0, 0, and the given number; this latter being now divided into periods of *three* figures each, the cube-root of the first period forms the first root-figure; this, as before, is used as a constant multiplier throughout the first step or stage of the operation, the products by it contributing to form *two* columns of work under the 0's, and a third column under the given number: thus, the root-figure is applied, as a multiplier, to the 1, and the product is added to the first 0; the same multiplier is applied to the sum, and the product added to the second 0; the same multiplier is applied to this next sum, and the product *subtracted* from the first period. To complete the step, we return to the 1, applying still the same multiplier, adding the product as before to the first column, applying the same multiplier to the sum, and adding the product to the second column, at which we *now* stop: we then return again to the 1; apply the multiplier, and add the product to the first column, carrying the process no further: the first step is now completed; the number last found, in the second column, is the trial-divisor for finding the next root-figure; with which root-figure we proceed through the same course of multiplications, &c., as at first, and thus get, under the trial, the true divisor, and thence the next dividend.

The following example exhibits the steps at length, the numbers in the three columns, which appear at the end of a step, being marked (1), (2), &c., as in the square-root form.

EXTRACTION OF THE CUBE-ROOT.

Ex. 1. Extract the cube-root of 411001037875.

1	0	0	411,001,037,875 (7435
	7	49	343
	—	—	—
	7	49	68001 (1)
	7	98	62224
	—	— (1)	—
	14	147	5777037 (2)
	7	856	4948407
	— (1)	—	—
	21	15556	828630875 (3)
	4	872	828630875
	—	— (2)	—
	214	16428	
	4	6669	
	—	—	
	218	1649469	
	4	6678	
	— (2)	— (3)	
	222	1656147	
	3	111475	
	—	—	
	2223	165726175	
	3		
	—		
	2226		
	3		
	— (3)		
	2229		
	5		
	—		
	22295		

You of course see why the figures below the (1), (2), &c. in the first column are put each one place further to the right; it is because the root-figures which produce them are each one place further to the right, the local value diminishing at a ten-fold rate. The corresponding numbers in the second column are pushed *two* places to the right, because the multiplicand and the multiplier from which each is produced have

both of them advanced *one* place to the right; and the figures are pushed *three* places to the right in the next column, because the multiplicand has advanced *two* places and the multiplier *one*.*

(102.) The foregoing process may be extended indefinitely. It may be applied to the finding of the fourth, fifth, sixth, or any higher root of a number: for the fourth root there will be four columns of work, for the fifth five columns, and so on.† But this is not the place to prosecute the subject further; for additional information you must consult the works referred to at the close of the foot-note below, as also for an explanation of the reason of the operation above, which cannot be made sufficiently intelligible without the aid of algebra. I shall give you one other example in the cube-root, in order that you may see how the work is to be contracted when the proposed number ends in a decimal not to be depended upon; or when only a prescribed number of decimals is required in the root. You will observe, by examining what follows, that the contractions are so managed as to exclude every decimal that would extend beyond the proposed limitation.

* Should any reader of this little work be already acquainted with arithmetic, but acquainted with it only as it is taught in the common school-books, he will be agreeably surprised to find that a subject which must have occasioned him so much perplexity as the extraction of the cube-root, resolves itself into the above simple process. I claim no merit for it myself; it is due to the late Mr. Horner, of Bath, as a particular application of his general method of solving numerical equations. One or two recent writers on arithmetic have had the discernment to see its superiority over the common rule, but not the generosity to mention, in connection with it, the name of its indefatigable author,—a man who has done more for the practical advancement of that part of calculation to which it belongs than any other mathematician since Newton. In order to prevent the possibility of this reproof being improperly applied, I must add,—though the addition is quite superfluous to those who know anything of his personal character,—that Professor De Morgan, who uses Horner's method in all his arithmetical writings, is never chargeable with this disingenuous oversight.

Students who may wish to know more about Horner's method, may consult my recent "Introduction to Algebra, and to the Solution of Numerical Equations;" "The Analysis and Solution of Cubic and Biquadratic Equations;" and "The Theory and Solution of Equations of the Higher Orders." This latter work, which, with the separately published "Appendix," costs 18*s*., is fit only for the advanced student.

† The fourth root may be obtained by first finding the *square*-root, and then the square-root of the result.

2. What is the cube-root of 98375·112? *

1	0	0	98375·112(46·163112	
	4	16	64	
	—	—	—	46·163112
	4	16	34375 (1)	21136164
	4	32	33336	———
	—	— (1)	———	18465245
	8	48	1039·112 (2)	2769787
	4	756	636·181	46163
	— (1)	———	———	27698
	12	5556	402·931 (3)	1385
	6	792	383·036	46
	———	——— (2)	———	5
	126	6348	19·895 (4)	1
	6	13·81	19·178	———
	———	———	———	2131·03300
	132	6361·81	·717	21136164
	6	13·82	·639	———
	——— (2)	——— (3)	———	852413200
	138	6375·63	78	127861980
	·1	8·30	64	2131033
	———	———	—	1278620
	138·1	6383·9\3	14	63931
	·1	8·3	13	2131
	———	——— (4)	—	213
	138·2	6392·2	1	4
	·1	·4		
	——— (3)	———	Proof...	98375·111
	1\3\8\·3	6\3\9\2\·6		———

To the above work I have annexed the reverse process of finding the cube of the root 46·163112, which, you see, may be depended upon as true, up to the sixth decimal place. In the first multiplication, which gives the square of the root, I have contracted the result to four places: in the next multiplication, which gives the cube, the factors are also arranged for four places of decimals; but, agreeably to the recommendation at page 126, only three of these places are preserved, in adding up the partial products. The result, you see, fully verifies the operation by which the root has been found; for, as the final remainder in that operation is 1, the number

* The decimal points in the several steps of the work *may* be omitted; but I think it safer to preserve them.

actually exhausted by the process is 98375·111. And you may, in general, consider the last decimal of a root determined in this manner as true to the nearest unit.*

(103.) I shall conclude with a few examples, for exercise, recommending you to use the contracted method, for decimals, as soon as you are familiar with the process in its uncontracted form. And now having conducted you to this point, I think I may presume that you are in full possession of every important principle in the elements of Arithmetic. The remaining few pages will be occupied with certain business-calculations, of too easy and obvious a character to render it necessary, with your present knowledge, that I should accompany them with the same minute details, and lengthened explanations, that I have furnished to you in what has preceded. If you have only made yourself completely master of what has now been taught, you may take up any more extensive treatise on Arithmetic, with the fullest confidence, that you will meet with no difficulty beyond your powers of successfully contending with. And, what is better, you will, I think, have acquired a deeper insight into the true principles of arithmetical calculation, than the majority of such treatises can afford you, inasmuch as you will have been habituated to think for yourself, to look for *reasons* as well as *rules*, and so have been fitted, by the proper mental preparation, to enter safely upon any department of *science* you may hereafter feel disposed to cultivate.

Exercises.

Find the cube-root of each of the following numbers.

1. 912673	7. 115·29736
2. 52734375	8. 822650
3. 21024576	9. 78314·6
4. 80677568161	10. 12345·678
5. 411001037875	11. 123·456789
6. 7835·8748	12. 9, to nine decimals.

* In marking off the figures to be dismissed in the contracted portion of the operation, it is of course matter of indifference whether dots be employed, as at page 130, or dashes, as in the example above. In working with the pen, the latter is the more convenient way; though it cannot be used in print without so separating the figures as to drive them out of the proper vertical columns. In print, therefore, dots would seem preferable; but when figures fall below those thus dotted, the dots might be mistaken for decimal points connected with the latter; so that, in the extraction of roots, it is better to tick off the figures as above, both in printing and in writing.

(104.) INTEREST, DISCOUNT, INSURANCE, &c.

Interest is the sum paid for the use of money by the borrower, or him who holds it, to the lender, or him who deposits it. The consideration agreed upon is usually at so much for the use of every £100 for a year, or, as men of business call it, at so much *per cent. per annum;* or, simply, at so much per cent., *per annum*, or per year, being understood.

The money lent or deposited is called the *Principal;* the consideration for £100 for a year, the *Rate* of Interest; and the principal, together with the interest, for any length of time, is called the *Amount* in that time: thus, if £100 be lent under an agreement that £5 is to be paid for the use of it for one year, then, £100 is the *principal* lent; £5 is the *rate* per cent.; £10 is the *interest* for 2 years; and £110 is the *amount* in that time.

The finding of the interest of a given sum of money at a given rate per cent. for 1 year, is obviously nothing more than a simple Rule-of-Three operation: for, as £100 is to the rate, so is the principal lent to the interest upon it; or, as £100 to the sum lent, so is the interest of £100 to the interest of the sum. And if the interest, thus determined for 1 year, be multiplied by any *number* of years, the product will be the interest accumulated in that time; or, instead of multiplying the interest for 1 year by the *number* of years, we may commence by multiplying the *principal* by that number, and make the *stating* afterwards. A formal Rule-of-Three stating is, however, generally dispensed with, and the operation conducted as follows.

(105.) *To find the Interest of a given Sum at a given Rate per Cent., for a given Number of Years.*

Rule. Multiply the principal by the number of years, the product by the rate, and divide the result by 100.

Note. When the rate is 5 per cent.,—a rate very commonly charged,—the operation, simple as it is, becomes still simpler: for the multiplication by 5, and the subsequent division by 100, may be replaced by a single division by 20; so that having multiplied the principal by the number of years, the 20th part of the product will give the interest.

Ex. 1. What is the interest of £587 16*s*. 4*d*. for 7 years, at 4, 5, and 6 per cent.?

£. *s*. *d*.	£. *s*. *d*.	£. *s*. *d*.
587 16 4	587 16 4	587 16 4
7	7	7
4114 14 4	2,0)411,4 14 4	4114 14 4
4		6
	205 14 $8\frac{1}{2}+\frac{2}{5}f.$	
164,58 17 4		246,88 6 0
20		20
11,77	5)205 14 $8\frac{1}{2}$, 5 per cent.	17,66
12	41 2 $11\frac{1}{4}$, 1 per cent.	12
9,28	164 11 $9\frac{1}{4}$, 4 per cent.	7,92
4	246 17 $7\frac{3}{4}$, 6 per cent.	4
1,12		3,68

Hence, the interest at 4 per cent. is £164 11*s*. $9\frac{1}{4}d.+\frac{8}{25}f.$; at 5 per cent., £205 14*s*. $8\frac{1}{2}d.+\frac{2}{5}f.$; and at 6 per cent., £246 17*s* $7^{3}d.+\frac{12}{25}f.$ The additional work in the middle column, shows how the interest at 4 and 6 per cent. may be obtained from that at 5 per cent., namely, by subtracting a fifth part of the latter interest, for the 4 per cent., and adding that fifth part for the 6 per cent. The fraction, $\frac{2}{5}$ of a farthing, has here been rejected: if it had been retained, the quotient by 5 would have been increased by $\frac{7}{25}f.$; so that the resulting interests for 4 and 6 per cent. would have been increased, the former by $\frac{8}{25}f.$, and the latter by $\frac{17}{25}f.$; and there would have been an exact agreement with the determinations of the other method.

I have been thus particular, more for the purpose of showing you the strict conformity between the results of different processes than because such minute accuracy is necessary in actual practice. Fractions of a farthing are, of course, always disregarded in business; and in calculations respecting interest, *pence* is in general the lowest denomination noticed. See art. 107.

2. Find the interest of £619 9*s.* 6*d.* for 1 year, at $5\frac{1}{2}$ per cent.

Either of the following three ways may be employed: in the last of these, the principal is halved, and the rate doubled.

	£.	*s.*	*d.*
	619	9	6
			5
	3097	7	6
$\frac{1}{2}$	309	14	9
	34,07	2	3
	20		
	1,42		
	12		
	5,07		

	£.	*s.*	*d.*	
	2,0)61,9	9	6	
	30	19	11,	int. 5 per cent.
$\frac{1}{10}$	3	1	$11\frac{1}{4}$,	at $\frac{1}{2}$ ditto.
	34	1	5,	at $5\frac{1}{2}$ ditto.

As $\frac{1}{2}$ is the tenth part of 5, the tenth part of the interest at 5 per cent. is added.

£.	*s.*	*d.*
309	14	9
		11
34,07	2	3
20		
1,42		
12		
5,07		

∴ £34 1*s.* 5*d.*=interest.

3. What is the interest of £500 for 4 years, at £5 7*s.* 6*d.* per cent.? The work is given below in four different ways.

Here, £5 7*s.* 6*d.* = £$5\frac{3}{8}$ = £5·375 or = $\frac{43}{8}$ £.

£500	£500	£500	£500
4	4	4	4
2000	2000	2000	2000
$5\frac{3}{8}$	5·375	43	5
10000	107,50·000	8)860,00	10000
750	20	£107 10*s.*	5*s.*=$\frac{1}{4}$ | 500
107,50	10,·00		2*s.* 6*d.*=$\frac{1}{8}$ | 250
20			107,50
10,00			20
			10,00

∴ £107 10*s.*=interest.

In the first of these methods we have to take $\frac{3}{8}$ of 2000, or, since $\frac{3}{8} = \frac{1}{4}+\frac{1}{8}$, we may take $\frac{1}{4}$ of 2000, which is the 500 above, and then add the half of that fourth, namely, 250, which gives 750. Instead of multiplying by 4 and by 5 in the first and third methods, we may, of course, multiply by 20 at once. The product of the *numbers* expressing the years and the rate may always be used instead of those

numbers separately, whenever it is thought convenient to do so, as in the next example.

4. What is the interest and amount of £212 10*s.* 4*d.* for $2\frac{3}{4}$ years, at $2\frac{1}{2}$ per cent.?

Here $2\frac{3}{4} = \frac{11}{4}$, and $2\frac{1}{2} = \frac{5}{2}$, $\therefore\ 2\frac{3}{4} \times 2\frac{1}{2} = \frac{55}{8} = 7 - \frac{1}{8}$.

	£.	*s.*	*d.*
	212	10	4
			7
	1487	12	4
$\frac{1}{8}$	26	11	$3\frac{1}{2}$
	14,61	1	$0\frac{1}{2}$
	20		
	12,21		
	12		
	2,52		
	4		
	2,10		

	£.	*s.*	*d.*
8)	212	10	4
	26	11	$3\frac{1}{2}$
$55 = 11 \times 5$			11
	292	4	$2\frac{1}{2}$
			5
	14,61	1	$0\frac{1}{2}$
	20		
	12,21		
	12		
	2,52		
	4		
	2,10		

	£.	*s.*	*d.*
	212	10	4
			$2\frac{3}{4}$
	425	0	8
$\frac{1}{2}$	106	5	2
$\frac{1}{4}$	53	2	7
	584	8	5
twice $2\frac{1}{2} =$			5
2)	2922	2	1
	14,61	1	$0\frac{1}{2}$
	20		
	12,21		
	12		
	2,52		
	4		
	2,10		

Hence the interest is £14 12*s.* $2\frac{1}{2}$*d.*, and, therefore, the amount is £227 2*s.* $6\frac{1}{2}$*d.*

(106.) *When the Interest for any Number of Days is required.*

Rule 1. As 365 is to the number of days, so is the interest for one year to the interest required; or, since twice 365 is 730, and since, moreover, in finding the interest for 1 year, we always have to divide by 100, after multiplying by the rate, we may proceed as follows:

Rule 2. Multiply twice the product of the principal and rate by the number of days, and divide by 73000.

Ex. 1. What is the interest of £325 10*s.* for 3 years and 89 days at 4½ per cent.?

```
                     By Rule 1.                              By Rule 2.
  £.  s.                    £.  s.  d.                  £.   s.   £.  s.
 325 10     365 : 89 : : 14 12 11¼                 2929 10 = 1464 15
      4½                    20                       89          × 2
 -------                 ------                  -------
1302  0                    292                    26361
 162 15                     12                   23432          s.
 -------                 ------                      44 10 = 10 × 89
14,64 15                  3515                   -------------£.  s.  d.
   20                        4           73 000)260 725 10(3 11  5¼
 -----                   -------                 219
12,95                    14062                   ----
  12                        89                    41725
 -----                   -------                     20
11,40                   126558                    -------
   4                    112496                    834,510(11
 -----                  -------                   73
 1,60              365)1251518(3428 298/365       --
                        1095                      104
                        -----                      73
                         1565                     -----
                         1460                     31510
                         -----    4)3429f. nearly.   12
£.  s.  d.               1051     ------          -------
14 12 11¼ + ⅗f. int. 1 yr. 730    12)857 1f.      378,120(5
        3                -----    ------          365
------------             3218     2,0) 7,1 5d.    -----
43 18 10¼ int. 3 yrs.    2920     --------         13120
 3 11  5¼ int. 89 days.  -----    £3 11  5¼            4
----------                298     ---------       -------
47 10  3½ whole int.                              52,480(1 nearly.
```

(107.) In this example, as well as in the examples which have preceded, I have beeu unnecessarily exact in computing the interest to the nearest farthing: but the odd farthings are disregarded in business-transactions, and interest calculations are considered sufficiently accurate when the true results are reached to the nearest penny. With this understanding, the operation by Rule 2 may be shortened. In dividing by so large a number as 73000, the odd shillings in the dividend may be disregarded; or, if above 10*s.*, the pounds may be increased by a unit. Now, since 73000 × *quotient* = *dividend*, it is plain, that if, by means of any division-operations upon 73000, we can reduce it to *unity*, the same operations upon the *dividend* will reduce it to the *quotient*, that is, to the interest for the proposed number of days. The number 73000 is reduced to unity, or, as nearly to unity as is necessary for the degree of accuracy

```
3)73000
  24333
   2433
    243
 -------
1·00009
 -------
```

required, thus: Divide it by 3; divide the result by 10, and then the result of this division by 10, in each case disregarding remainders. Add up the four numbers, as in the margin, and point off five places for decimals. The result is 1·00009, which differs from unity by a quantity too insignificant to deserve notice. These operations, therefore, performed upon the *dividend*, will give the interest with all necessary accuracy. Let us, for instance, apply them to the dividend 260730, exceeding that above by £4 10*s*., the 73000th part of which is of no moment. You see that the result is £3 11*s*. 5¼*d*., as before; and that it differs from the strictly accurate conclusion above only by a fraction of a farthing.

3)260730£
86910
8691
869

3·57200£
20

11·440 *s*.
12

5·28 *d*.
4

1·12 *f*.

It is this method which I would recommend you always to adopt in computing interest for days.

I here give another example of it, which, for variety, is worked in the margin by decimals.

2. What is the interest of £956 14*s*. 6*d*. for 7 days, at 4½ per cent.?

Here, 14*s*. 6*d*. is reduced to the decimal of a £, and the product by the days multiplied by double the rate: the decimal of a £, in the resulting product, is rejected, and the integral value increased by unit. The answer is £0 16*s*. 6*d*.

NOTE. When the interest is 5 per cent., then, since the double of the rate is 10, we have only to multiply by the number of days, and to point off but *four* places from the result of the remaining operations.

£956·725
7

6697·075
9

60273·675
3)60274
20091
2009
200

·82574£
20

16·5148 *s*.
12

6·1776 *d*.

(108.) You perceive that the particulars concerned in interest-calculations are these four, namely, *Principal*, *Rate*, *Time*, and *Interest*; the *Amount* is merely the sum of two of these, the principal and the interest. In the preceding examples, the three of these which have been given to find the fourth are the *first* three; but any other three being given, the fourth may be found, simply by reversing one or more of the operations exhibited above.

Ex. 1. In what *time* will £91 13*s.* 4*d.* amount to £105 6*s.* 0½*d.* at 4¼ per cent.?

The principal and amount being given, the *interest* is given, namely, £13 12*s.* 8½*d.*; and we want to know the number of *years* which has produced this amount of interest. For this purpose it is clearly only necessary to divide £13 12*s.* 8½*d.* by 1 year's interest.

£.	*s.*	*d.*		£.	*s.*	*d.*
91	13	4		13	12	8½
		4¼		20		
366	13	4		272		
22	18	4		12		
389	11	8		3272		
20				2		
7791			1870)	6545	(3½ years. Ans.	
12				5610		
935,00 *d.*, 1 year's interest.				935		
2				935		
1870, halfpence.						

2. What *principal* put out to interest for 3½ years, at 4¼ per cent., will amount to £105 6*s.*?

For a principal of £100, the amount in the given time at the given rate is £100 + £4·25 × 3·5 = £114·875. Consequently, since £105 6*s.* = £105·3

114·875 : 105·3 :: £100 : £91 13*s.* 4*d.* Ans.

	£.	£.
1,1,4,·8,7,5)	10530	(91·667
	103388	20
	1912	13·34 *s.*
	1149	12
	763	4·08 *d.*
	689	
	74	
	66	
	8	
	8	

3. What must be the *rate* per cent. in order that £142 10*s.* may amount to £163 13*s.* 11*d.* in $4\frac{1}{4}$ years?

Here the interest is £21 3*s.* 11*d.*, which is what would arise from multiplying £142 10*s.* by $4\frac{1}{4}$, by the *rate*, and dividing by 100: consequently, we have only to divide £142 10*s.* by 100, to multiply the quotient by $4\frac{1}{4}$, and then to divide £21 3*s.* 11*d.* by the result, in order to get the *rate*.

£.	*s.*
142	10
	20
2850	
12	
342,00	
$4\frac{1}{4}$	
1368	
$85\frac{1}{2}$	

£.	*s.*	*d.*
21	3	11
20		
423		
12		
5087		
2		

$1453\frac{1}{2} \times 2 = 2907$)10174($3\frac{1}{2}$, rate.
8721
————
1453
1453
————

There can be no sensible error in the rate by making the amount £163 14*s.* instead of £163 13*s.* 11*d.*; in which case the interest is £21 4*s.* = £21·2: we may therefore work thus by decimals:—

£1·425 = principal ÷ 100
————
5·700
·35625
———— £.
£6,·0,5625)21·2(3·5, rate.
182
————
30
30
————

In the second method here given, all those decimals of the divisor have been cut off which, upon multiplication by the quotient-figure, would fall to the right of the decimal ·2 in the dividend, agreeably to what has been taught in contracted division, because the decimal ·2 is not strictly accurate, although the error is very minute.

Exercises.

1. What is the interest of £9826 13*s.* 8*d.* for 1 year, at $2\frac{1}{2}$ per cent.?
2. What is the interest of £896, for $2\frac{1}{2}$ years, at $3\frac{1}{4}$ per cent.?
3. What is the interest of £98 19*s.* 6*d.* for 11 months, at $3\frac{3}{4}$ per cent.?
4. What is the interest of £3204 14*s.* for 37 days, at 5 per cent.?

5. What is the interest of £256, from May 7 till Aug. 12, at $4\frac{1}{2}$ per cent.?
6. What is the interest of £319 0*s.* 6*d.* for $5\frac{3}{4}$ years, at $3\frac{3}{4}$ per cent.?
7. What is the amount of £120, from Jan 7 to Sept. 12, 1852 (leap year) at 4 per cent.?
8. What principal will produce a yearly interest of £341 5*s.* at 5 per cent.?
9. In what time will £2000 amount to £2280, if lent at $3\frac{1}{2}$ per cent.?
10. If £42 3*s.* 9*d.* be received for interest on £11250 for 1 month, what is the rate per cent. per annum?
11. What is the interest of £193 12*s.* at £11 18*s.* 6*d.* per cent.?
12. The amount of money expended for the maintenance of the poor by the 607 unions of England and Wales, during the year ending at Michaelmas 1850, was £3469857, and during the year ending at Michaelmas 1851 the amount was £3288192; what was the decrease per cent.?
13. The population of Great Britain in 1841 was 18664761, and in 1851 it was 20936468: required the increase per cent. during the intervening ten years.
14. The population of Ireland in 1841 was 8175124, and in 1851, 6515794: required the decrease per cent.

(109.) *Discount*, in the usual meaning of the term, is only another name for *Interest.* In commercial transactions, payments are not always made in *money*, but often in what are called *Bills of Exchange* or *Promissory Notes.* These are stamped slips of paper, on which engagement is made to pay in cash, after the lapse of a specified time. The *present worth* of such a Bill, is that sum of money, paid down, which, when put out at the proposed interest for the specified time, will amount to just sufficient to pay the Bill when it becomes due. Thus, if the Bill be for £105, payable in one year, interest being at 5 per cent., then, since £100 present money would amount in one year to £105, if placed out at the proposed interest, £100 is the present worth of the Bill; the £5 thus allowed for *cashing* it being the true deduction or *Discount.* But Bankers do not discount Bills on these terms: it is not reasonable to expect they should: they must make *a profit* by this as well as by other departments of their

business, and, therefore, they would charge, as *discount*, the full *interest* of the £105 for one year, namely, £5 5*s.*; so that by discount, commercial men understand *interest upon the sum discounted*, during the period for which the payment of that sum is delayed; what remains, after the deduction of this discount, is all that is paid down for a Bill, as its *present worth*; hence, to calculate the discount of a sum of money, is the same as to calculate the interest of that sum, the time and rate per cent. being given. In Bills, however, three days, called *days of grace*, are added to the time specified for payment:* thus, if a Bill be drawn for three months after date, and be dated on the 1st of January, it does not become due till the 4th of April; and the interest or discount is accordingly calculated for three months and three days. In general, the whole time the Bill has to run is turned into days, and the interest computed as at page 160. The following Table will be found very convenient in calculations of this kind.

Table for the Number of Days from any Day in one Month to the same Day in another.

Remember that in Leap Year another day is to be added to February.

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
January	365	31	59	90	120	151	181	212	243	273	304	334
February	334	365	28	59	89	120	150	181	212	242	273	303
March ..	306	337	365	31	61	92	122	153	184	214	245	275
April ..	275	306	334	365	30	61	91	122	153	183	214	244
May....	245	276	304	335	365	31	61	92	123	153	184	214
June....	214	245	273	304	334	365	30	61	92	122	153	183
July....	184	215	243	274	304	335	365	31	62	92	123	153
August..	153	184	212	243	273	304	334	365	31	61	92	122
Septemb.	122	153	181	212	242	273	303	334	365	30	61	91
October..	92	123	151	182	212	243	273	304	335	365	31	61
Novemb.	61	92	120	151	181	212	242	273	304	334	365	30
Decemb.	31	62	90	121	151	182	212	243	274	304	335	365

* The reason of this is, that the law allows the indulgence of *three days* to the *acceptor* of a bill, as the person on whom it is drawn is called, before legal proceedings can be issued against him for non-payment; but the bankers take care that the "indulgence" shall be paid for. The acceptor becomes legally responsible for the payment of the bill by simply writing his name across it, by doing which he is said to *accept* it.

Ex. 1. A Bill for £77, drawn on the 8th of March, at 6 months, is discounted on the 3rd of June, at 5 per cent.: required the discount.

The 6 months expire on the 8th of September; therefore the Bill becomes *due* on the 11th of September. From the 3rd of June to the 3rd of September is 92 days, and, therefore, to the 11th of September it is 100 days. The interest of £77, at 5 per cent., for 100 days, is found, by the method already taught, to be £1 1*s.* $1\frac{1}{4}$*d.*, the discount required.

The following example is one belonging to a class of cases of frequent occurrence in business.

2. A Bill for £500 was due Feb. 2, 1851: of this sum, £80 was paid, March 9; £115, May 15; £25, June 1; and the balance, namely £280, Aug. 14: what interest was due at 5 per cent.?

		£	£
1851, Feb. 2	Due 500 × 35	= 17500	3)71545 *See p.* 160.
Mar. 9	Paid 80		23848
			2384
	420 × 67	= 28140	238
May 15	115		
			£9·8015
	305 × 17	= 5185	20
June 1	25		
			∴ £9 16*s.* Int. due.
	280 × 74	= 20720	
Aug. 14	280		
		71545	

Exercises.

1. What is the present worth of a Bill drawn on the 10th of January, 1852 (leap-year), at three months, for £1264 11*s.* 8*d.*, at 4 per cent.?
2. How much cash must be received for a Bill for £218 11*s.* 8*d.*, drawn on the 14th of August, at 4 months, and discounted on the 3rd of October, at 4 per cent.?
3. How much must be received for a Bill for £568 12*s.* 9*d.*, dated April 27, at 7 months, and discounted June 3, at 5 per cent.?
4. What is a Bill for £1570 10*s.* 6*d.* worth on the 10th of January, supposing it to have been drawn at 5 months on the 30th of the preceding December: interest $3\frac{1}{2}$

5. A Bill for £39 5*s.* falls due on the 1st of September, but payment is offered on the 3rd of July preceding: what cash should be paid, the discount being at 5 per cent.?
6. A Bill for £150, drawn 11th of July, at 3 months, was discounted Sept. 1st, at $5\frac{1}{2}$ per cent.: how much did the holder of it receive?
7. A Bill for £500 was due Feb. 2, 1851, of which £80 was paid on the 9th of March following; £115 on the 15th of May; £25 on the 1st of June; and the balance on the 14th of August: how much interest at 5 per cent. was due?

NOTE. The present worth of the bills in the foregoing examples is what remains after deducting the *banker's discount*, which, as you have already been told (page 163), includes his profit, and is more than the true discount. The rule for *this* is as follows:—

As £100 increased by the interest for the time, that is, as the *amount* of £100 is to £100, so is the *amount* of the bill to its true present worth; as is obvious.

If bills were drawn at a very long date, the banker's discount would be enormous. Thus, a bill of £100 at 20 years, allowing 5 per cent. (the usual rate), would produce *nothing:* for the interest, or banker's discount, would be just £100. And if it were not made payable till 40 years, the holder of it would have to give £100 to the banker to take it off his hands! The bankers' principle, therefore, when applied to such long periods, is manifestly unjust and absurd; but as bills are generally made payable within a few months, the banker's discount exceeds the true discount by no more than what may be considered a reasonable profit on the transaction. It must be remembered, too, that the discounter runs *some risk*, so that although long bills are apparently more profitable than short ones, yet bankers are less inclined to discount the former than the latter.

(110.) BROKERAGE, COMMISSION, INSURANCE, &c.

COMMERCIAL and money transactions are seldom conducted on a large scale, except through the agency of a third party, who is paid so much per cent. for his services. The sum of money engaged in the transaction, and the agent's per-centage being given, the whole allowance to the agent is, of course, computed in the same way as interest is computed, from which, indeed, it differs only in name, and in being generally free from considerations of *time*. If goods or merchandise be bought or sold through an agent or *factor*, the per-centage on the money is called *Commission.* If money be employed through an agent in discounting Bills, in the purchase of

Shares in a trading company, or in the public funds or *Stocks;* or, if an agent dispose of such money interests for another, the per-centage he receives is called *Brokerage;* and he himself is a Bill-broker, a Share-broker, a Stock-broker, &c.

Insurance is the name given to what is paid to an Insurance Office, at the rate of so much per cent. on the value of property exposed to loss by Fire, Shipwreck, &c. The parties who agree to compensate for the loss, are the *Insurers* (or, in the case of ships, *Underwriters*, from their undersigning the agreement); the party protected is the *Insured;* the money paid for the protection, the *Premium;* and the parchment, which binds the parties to the contract, the *Policy.* Life Insurance, or Life Assurance, is of a similar nature: it secures the payment of a specified sum when the *assured* dies, upon his paying so much per cent. per annum on that sum during his life, or a sum down, equivalent to the *annual premium.* Whatever be the name given to the agent employed, or to the service performed, it is plain, that the allowance on a specified sum of money, at a specified rate per cent., can involve no calculations different from those which come under the head of *Interest:* a few examples, therefore, will be all that need be given here, special rules being unnecessary.

Ex. 1. A factor sells merchandise to the amount of £2575 17*s.* 6*d.*, and charges 4*s.* per cent. for commission: how much is to be paid him?

£.	*s.*	*d.*
25,75	17	6
	20	
15,17		
	12	
2,10		

	£.	*s.*	*d.*
5)	25	15	2
	£5	3	0½. Ans

Here, the interest, or commission, at 1 per cent. is £25 15*s.* 2*d.*; therefore, that at 4*s.* per cent. is a fifth part of this, or £5 3*s.* 0½*d.*

In such small per-centages it is not worth while to take account of the odd pence in the sales; so that dividing £2575 17*s.* by 5, for the ⅕ per cent., and disregarding the pence in the quotient, we may proceed as in the margin.

	£.	*s.*
5)	2575	17
	5,15	3
		20
		3,03

∴ £5 3*s.* = Commission.

In the next example, too, the 9*d.* may be omitted in finding the broker's charge, which would be 14*s.* 5*d.*

2. How much money will purchase £575 10*s.* Bank Stock, worth 131¾ per cent., and how much must be paid to the Stock-broker, who charges ⅛ per cent., that is, 2*s.* 6*d.* per cent., on the stock purchased?

```
   £.   s.                    £.   s.  d.
 4)575  10                    575  10
         8                    182  14   5
 ---------                   -----------
  4604   0                   £758   4   5
         4                   -----------
 ---------
 18416   0                     £.   s.  d.
   143  17   6*÷2 =,71         18   9
 -------------                 20
 182,72  2   6                ----
    20                        14,38
  ----                           12
  14,42                       ----
     12                        4,65
  ----                            4
   5,10                       ----
                               2,60
```

Here, to find the purchase-money, it is plain, that we shall only have to add to £575 10*s.*, the *interest* of it at 31¾ per cent.: this interest will be best found by taking 32 per cent., and deducting ¼ per cent., as in the margin: the purchase-money is thus found to be £758 4*s.* 5*d.*, and the broker's claim, 14*s.* 4¾*d.*, which is ⅛ of £575 10*s.*, divided by 100.

NOTE. Those who purchase stock, purchase nothing more than a claim to a certain amount of interest, payable half-yearly. Stocks are of different kinds, not only as to the names they bear, but as to the rate of interest paid to the purchaser. If stock yield a high rate of interest, it is proportionally dear; and more than £100 in money must be given for £100 stock, when the interest the latter yields is more than can be got for £100 in money. Although property in stock cannot be *taken* out, it can always be *sold* out with but little trouble. The prices of stock, of whatever kind, varies from time to time, sometimes rising and sometimes falling, like other purchasable commodities; the fluctuations, however, are usually inconsiderable, except when the tranquillity of the country is in danger; funded property then becomes insecure, and stocks fall in proportion to the alarm.

3. A cargo, worth £3800, is to be insured at 5 per cent.: for what amount must the insurance be effected, so that, in case all should be lost, the owners may recover both the value and the premium paid?

For an insurance for £100, a premium of £5 must be paid; this being deducted, leaves £95; so that an insurance for £100 can cover a loss of only £95: therefore, to find what insurance can cover a loss of £3800, we have the proportion, £95 : £100 :: £3800 : £4000, the amount to be

* The half of this is ⅛th of £575 10*s.*

insured. And in this way is the insurance to cover loss and premium always to be found; namely, as £100 diminished by the rate : £100 :: value of the property : the sum to be insured. If there be any other per-centage, as for commission, policy, &c., it must be deducted from £100 in the same way.

Exercises.

1. What is the commission on £3698 12*s.*, at $3\frac{1}{2}$ per cent.?
2. If a person sell out £600 stock, when the price of it is $83\frac{3}{8}$ per cent., how much will he receive after paying $\frac{1}{8}$ per cent. on the stock sold for brokerage?
3. What amount must be insured to cover £1880, together with the premium of £5 5*s.* per cent., 5*s.* per cent. for the policy, and $\frac{1}{2}$ per cent. for commission?
4. What amount must be insured on £1938 12*s.* 6*d.* at $5\frac{3}{4}$ per cent., so as to provide also for the premium?
5. What is the commission on £876 5*s.* 10*d.* at $3\frac{3}{4}$ per cent.?
6. What is the brokerage on £372 7*s.* 4*d.* at 4*s.* 6*d.* per cent.?
7. How much must be given for £912 14*s.* stock, at $127\frac{3}{4}$ per cent.?
8. Required the brokerage on the purchase of £11675 17*s.* stock, at $\frac{1}{8}$ per cent.
9. What will it cost to purchase £7391 14*s.* 9*d.* stock, the price being $86\frac{1}{4}$, and the brokerage $\frac{1}{8}$ per cent.?
10. Find the expense of insuring a cargo worth £850, at £2 12*s.* 6*d.* per cent., policy duty 5*s.* per cent., and commission $\frac{1}{2}$ per cent.

(111.) COMPOUND INTEREST.

All the preceding calculations respecting interest proceed on the supposition that the interest is actually paid when due. If, however, the interest be withheld for any time, then this interest so withheld becomes a new principal, and itself produces interest. The whole interest thus accumulated in any time is called *compound interest;* while that which arises solely from the original principal, and which has been the subject of the preceding articles, is, for distinction, called *simple interest.* An example will be sufficient to show you how compound interest may be calculated; but as the work, though made up of very easy and obvious steps, becomes

tedious and lengthy when those steps are numerous, tables have been contrived to save the labour. To these I must refer you for expeditious calculation; but the example worked in the margin, namely, to find the compound interest of £550, at 5 per cent., when the simple interest, which should be paid yearly, is withheld or *forborne* 4 years, will fully put you in possession of the mode of proceeding without tables. The simple interest for the first year, computed in the usual way, instead of being

£.	s.	d.	
20)550	0	0	
27	10	0	int. 1st yr.
20)577	10	0	amount.
28	17	6	c. int. 2nd yr.
20)606	7	6	amount.
30	6	$4\frac{1}{2}$	c. int. 3rd yr.
20)636	13	$10\frac{1}{2}$	amount.
31	16	$8\frac{1}{4}$	c. int. 4th yr.
668	10	$6\frac{3}{4}$	amount.
550	0	0	original prin.
118	10	$6\frac{3}{4}$	compd. int.

paid, is added to the principal; the amount is the principal for the second year, the interest of which is the compound interest of the original principal for the second year, and so on till the expiration of the 4th year, when the amount becomes £668 10*s*. $6\frac{3}{4}$*d*., which, diminished by the original principal, leaves £118 10*s*. $6\frac{3}{4}$*d*. for the whole compound interest:—the same sum that we should get by adding the interest for the several years together. The simple interest of the proposed sum would be 4 times £27 10*s*., or £110; so that the difference is £8 10*s*.

(112.) PROPORTIONAL PARTS.

MANY useful and interesting questions depend for their solution upon the division of quantities into parts, having specified ratios to one another. I shall here give you a short article on the mode of effecting this division.

To divide a Quantity into Parts, such that any one Part shall be to another, as one given Number to another.

RULE. As the sum of the given numbers, is to any one of them, so is the whole quantity to be divided, to the part corresponding to that number.

For example: suppose it were required to divide £80 into three parts, that should bear to one another the same relations as the numbers 2, 3, and 5. The parts would be found as follows. 10 : 2 :: £80 : £16; 10 : 3 :: £80 : 24; 10 : 5 :: £80 : £40.

The required parts are therefore £16, £24, and £40; which together make up the £80, and which obviously bear the proposed relation to one another; namely, £16 : £24 :: 2 : 3; £16 : £40 :: 2 : 5; £24 : £40 :: 3 : 5. It is scarcely necessary to tell you, that when all the parts but one are thus found by proportion, that one may be got by subtracting the sum of all the others from the *whole*.

The principle of the rule can scarcely require any explanation to a person familiar with proportion: the *sum* of the parts of a quantity, and the *sum* of the parts of a number, are given; the subdivisions of the *number* are also known; and, since the whole of anything is to a part, as the whole of another thing to a *like* part, the sufficiency of the rule is obvious.

Exercises.

1. Three traders, A, B, and C, contribute the following sums to the business: A, £500; B, £650; and C, £700: the year's profits are £555. Required each person's share of them.
2. Gunpowder is composed of nitre, charcoal, and sulphur, thus: nitre, 76 parts; charcoal, 14; sulphur, 10: how much of each is used in 1 cwt. of gunpowder?
3. How much pure gold, and how much alloy, are contained in a guinea? (See p. 39.)
4. Standard silver contains 37 parts of pure silver, and 3 of copper: how much of each ingredient is there in £1 7*s.* 6*d.*; 1 lb. troy being coined into 66 shillings?
5. 100 lbs. of pure water contain 88·9 parts of the gas called oxygen, and the remaining 11·1 parts of the gas called hydrogen: what weight of each is there in a cubic foot, or 1000 oz. of water?
6. A bankrupt owes A. £120; B, £80; and C, £75: he possesses £165, which he is anxious to divide equitably: how much should each creditor receive?
7. Pewter is composed of 112 parts of tin, 15 of lead, and 6 of brass: how much of each enters into the composition of 1 ton of pewter?

8. A person bequeathed in his will, £140 to A; 100 guineas to B; 80 guineas to C; £70 to D; and £60 to E: but, at his death, the whole amount of his property was but £311 15*s.*: how much of this should A, B, C, D, and E. receive?

(113.) THE CHAIN RULE.

The Chain-Rule is a compendious rule for working examples which involve several Rule-of-Three statings; and by which, questions in Compound Proportion may be otherwise briefly solved. The following is an example of it.

If 3 lbs. of tea be worth 8 lbs. of coffee, and 5 lbs. of coffee worth 18 lbs. of sugar, how many lbs. of sugar should be exchanged for 20 lbs. of tea?

Here, by two simple statings,

we have 8 lbs. coffee : 5 lbs. coffee :: 3 lbs. tea : $\frac{3\times 5}{8}$ lbs. tea;

and $\frac{3\times 5}{8}$ lbs. tea : 20 lbs. tea :: 18 lbs. sugar : $\frac{8\times 18\times 20}{3\times 5}$ lbs. sugar; but, by the chain-rule, the particulars would be arranged thus:

3 lbs. tea = 8 lbs. coffee,
5 lbs. coffee = 18 lbs. sugar,
how many lbs. sugar = 20 lbs. tea?

where no two commodities of the same kind occur in the same column; and then, by dividing the product of the numbers in the *complete* column by the product of those in the column which the *answer* would make complete, the number which expresses the answer is obtained; namely, $\frac{8\times 18\times 20}{3\times 5}$ $= 192$, $\therefore$ 192 lbs. sugar is the answer.

Exercises.

1. If 3 lbs. of pepper be worth 4 lbs. of mustard, and 5 lbs. of mustard be worth 12 lbs. of candles, how many pounds of candles should be exchanged for 20 lbs. of pepper?
2. If 5 lbs. of tea be worth 12 lbs. of coffee, and 9 lbs. of coffee worth 28 lbs. of sugar, and 13 lbs. of sugar worth 18 lbs. of soap, how many lbs. of soap may be had for 7 lbs. of tea?

3, £1 sterling = 420*d.* Flemish, 58*d.* Flemish = 1 crown of Venice, 10 crowns Venice = 6 Venetian ducats, 1 ducat = 360 mervadies Spanish, and 272 mervadies = 1 Spanish piastre: how many piastres = £1000 sterling?

The above rule is chiefly used in questions like this last, relating to exchanges with foreign countries; numerous applications of it, to matters of this nature, are given in Kelly's Universal Cambist, vol. ii.

(114.) DUODECIMALS.

You already know that the common notation of Arithmetic is called the *decimal notation*, because the local value of every figure of a number expressed in that notation diminishes at a *ten-fold* rate, as it is removed from place to place towards the right. If the diminution were at a *twelve-fold* rate, the notation would be the *duodecimal notation.* In the measurement of *lengths*, the denominations feet and inches do actually descend in value, in this way:—an inch being the twelfth part of a foot; so that, for the purposes of *Mensuration*, it is convenient to have a *duodecimal arithmetic*, at least for the operation of multiplication.

In the decimal notation, 16·3, means 16 and 3 *tenths:* if these were twelfths instead of tenths, we might write the number thus: 16 3, leaving a gap between the two denominations; and, in multiplying this by any number in the same notation, it is plain, that we may proceed just as with decimals, provided we take care to carry *twelves* instead of *tens:* thus, to multiply 16 3 by 7 9, we should say, 9 times 3 are 27; 3, and carry 2 *twelves:* 9 times 16 are 144, and 2 are 146; which is 12 *twelves*, and 2: 7 times 3 are 21; 9, and carry 1: 7 times 16 are 112, and 1 are 113: so that the product is 125, 11 *twelfths* of one of the units in the 125, and 3 *twelfths* of one of the units in the 11. If we were to work by common arithmetic, we should, of course, get the same thing: thus, $16\frac{3}{12}\times 7\frac{9}{12} = \frac{195}{12}\times\frac{93}{12}$; and this operation performed, as in the margin, gives the above result, namely, $125\frac{11}{12}+\frac{3}{12}$ of $\frac{1}{12}$.

$$\begin{array}{rrr} 16 & 3 & \\ & 7 & 9 \\ \hline 12 & 2 & 3 \\ 113 & 9 & \\ \hline 125 & 11 & 3 \\ \hline \end{array}$$

$$\begin{array}{r} 195 \\ 93 \\ \hline 585 \\ 1755 \\ \hline 12)18135 \\ \hline 12)\ 1511\frac{3}{12} \\ \hline 125\frac{11}{12} \end{array}$$

In our multiplicand, 16 3, above, the 16 might have been the *number* of feet in the length of a floor or wall, and the 3, the number of inches besides; and the multiplier, 7 9, might have been, in like manner, the *number* of feet and inches in the width or height: the first principles of Mensuration show us, that the product, namely, 125 11 3, would be the *number* of square feet, twelfths of a square foot, and twelfths of a twelfth, that is, $\frac{1}{144}$ths, or square inches, in the *surface* of that floor or wall. The twelfths of a square foot are called *parts;* so that the measure of the surface in question would be 125 sq. ft., 11 pts., 3 sq. in. Suppose, for instance, it were required to find the measure of a ceiling, which is 22 ft. 5 in. long, and 16 ft. 11 in. wide: a workman would compute the surface, as in the margin, using the number in the multiplier, connected with the higher denomination, *first*, instead of second, as above; and he would thus find the surface to contain 379 sq. ft., 2 pts., 7 sq. in. (See the remarks at p. 62.)

ft.	*in.*	
22	5	
16	11 .	
138	8	
22		
20	6	7
379	2	7

Exercises.

Find the *surface-measures* from the following *linear measures.*

1. Length, 32 ft. 9 in.; breadth, 8 ft. 3 in.
2. Length, 20 ft. 6 in.; breadth, 17 ft. 9 in.
3. Length, 65 ft. 10 in.; breadth, 29 ft. 6 in.
4. Length, 97 ft. 9 in.; breadth, 16 ft. 6 in.
5. Length, 75 ft. 9 in.; breadth, 17 ft. 7 in.
6. Length, 97 ft. 8 in.; breadth, 8 ft. 9 in.
7. Length, 59 ft. 6 in.; breadth, 3 ft. 11 in.
8. Length, 87 ft. 5 in.; breadth, 35 ft. 8 in.

Examples such as these belong more properly to Mensuration. I give them here, as a conclusion to this Rudimentary Treatise on Arithmetic, in order that you may have a little experience, before commencing the subject just named, in what is commonly, but improperly, called the multiplication of length by length, or by width; and, at the same time, be furnished with a practical illustration of the remarks at p. 62.

MISCELLANEOUS EXAMPLES.

1. What decimal of a £ is £$\frac{31}{758}$, and what fraction of a shilling is ·625*s.*?
2. What fraction of a half-crown is $2\frac{1}{12}$*s.*?
3. What is the interest of £400 at $3\frac{1}{2}$ per cent. for $2\frac{1}{2}$ years and 59 days?
4. What is the interest of £712 6*s.* for 8 months, at $7\frac{1}{2}$ per cent.?
5. Required the price of 73 lbs. 5 oz. 15 dwt. of silver, at 5*s.* 9*d.* per oz.
6. How much must be paid for the carriage of 8 cwt. 3 qrs. 7 lbs., if 10*d.* be charged for 1 stone of 14 lbs.?
7. How many men will be sufficient to dig a trench 135 yards long in 8 days, when it is known that 16 men can dig 54 yards of it in 6 days?
8. What is the price of 7985 articles at 7*s.* $10\frac{1}{4}$*d.* each?
9. What does the commission on £530 2*s.* 9*d.* amount to at 2*s.* 6*d.* per cent.?
10. What is the worth of £4563 10*s.* Bank Stock, at $127\frac{3}{4}$ per cent.?
11. If 5 cwt. 3 qr. 14 lbs. be bought for £9 8*s.*, and sold for £11 18*s.* 11*d.*, what is the gain per cwt.?
12. If the prime cost of an article be at the rate of £4 16*s.* per cwt., how much per cwt. must it be sold at to produce a gain of 15 per cent.?
13. What sum of money will produce as much interest in $3\frac{1}{4}$ years as £210 3*s.* will produce in 5 years and 5 months, at the same rate per cent.?
14. At the battle of Bunker Hill, in America, the roar of the cannon was so distinct at Hanover, 120 miles distant, that business was for a time suspended there; how long was the sound in travelling?
15. A cistern holding 400 gallons is supplied by a pipe at the rate of 7 gallons in 5 minutes, but a gallon leaks out every 3 minutes; in what time will it be full?
16. What is the least common multiple of 12, 26, 56, and 182?
17. At what time between 9 and 10 o'clock are the hour and minute hands of a watch exactly together?
18. Find the square-root of 29929, and the cube-root of 228099131.

19. Required the values of $\sqrt[3]{78314}$ and $\sqrt{\frac{45}{64}}$.
20. How long will it take to count a million, at the rate of 100 a minute, for 10 hours a day?
21. If 56 lbs. of bread be sufficient for 7 men 14 days, how much will serve 21 men 3 days?
22. Find the values of 73^3, 27^4, and $\sqrt[4]{5719140625}$.
23. If $\frac{3}{8}$ oz. avoirdupois cost $\frac{7}{8}$*s.*, what will $\frac{5}{6}$ lb. cost?
24. How much Stock, at $111\frac{3}{8}$ per cent., can be purchased for £10000?
25. What decimal of 1 lb. avoirdupois is 7 drams?
26. What difference is there between the *Banker's* discount of £350, at 4 per cent., for 8 years, and the *true* discount of the same sum, for the same time, and at the same rate per cent.?
27. For what sum must an insurance be effected, in order to cover a loss of £3500, together with the premium of £3 8*s.* per cent.?
28. What is the interest of £51·425, at 4·125 per cent.?
29. What sum of money will produce £·35 interest in $1\frac{3}{4}$ year, at 4 per cent.?
30. A Bill of £170 was due August 12, and left unpaid till Sept. 18, when £54 was paid: the balance remained till Oct. 17, when £56 was paid; and, finally, the whole was settled Nov. 14: how much interest was due at 5 per cent.?
31. How much sugar, at $4\frac{1}{2}$*d.* per lb., ought to be given in exchange (in *barter*, as it is called) for 17 cwt. of cheese, at £3 10*s.* per cwt.?
32. If 7 gallons of brandy be worth 9 gallons of rum, and 9 gallons of rum worth 12 gallons of geneva, what is the price of a gallon of each separately, when the three gallons together cost £2 2*s.* 6*d.*?
33. What is the true present worth of £357 10*s.*, due 9 months hence, interest being 5 per cent.?
34. What are the values of $\sqrt{27\frac{9}{16}}-\sqrt{9\frac{48}{49}}$, and $\sqrt{\frac{2704}{4225}}-\sqrt{\frac{2304}{5184}}$?
35. Required the values of $\sqrt[3]{\frac{1}{2}}$ and $\sqrt[3]{7\frac{1}{5}}$ to six decimals.
36. A person has but £100 to pay the following sums; namely, to A, £48 15*s.*; to B, £72 10*s.*; and to C, £84 13*s.* 4*d.*: what share ought to be paid to each?
37. If A can do a piece of work in 10 days that B can do in 13, in what time can they do it together?

38. How many square yards are there in a piece of ground 864 ft. 3 in. long, and 62 ft. 6 in. wide?
39. What is the premium of insurance on £675 11*s.* 8*d.* at £5 13*s.* 9*d.* per cent.?
40. Two persons, A and B, enter into partnership; the stock of A, £280, is employed for 5 months, and that of B, £266 13*s.* 4*d.*, for six months; the profits are £331 12*s.* 6*d.*: divide them equitably.*
41. Three persons, A, B, and C, trade in concert; A contributes £89 5*s.* for 5 months; B, £92 15*s.* for 7 months; and C, £38 10*s.* for 11 months; how should the profits, namely, £86 16*s.*, be divided?
42. Multiply ·134786 by ·288793 to as many places as can be depended upon; the last decimal in each factor being only approximately true. Also, divide 14 by ·7854, the last decimal being only approximately true.
43. How much tin and copper is a bell of 150 lbs. composed of, there being in it three times as much copper as tin?
44. Proof spirits contain 48 parts of pure spirit, and 52 of water; how much of each is there in 84 gallons of proof spirits?
45. At what rate per cent. will £956 amount to £1314 10*s.* in $7\frac{1}{2}$ years, simple interest?
46. If 3 lbs. of tea be worth 7 lbs. of coffee, and 13 lbs. of coffee worth 48 lbs. of sugar, and 15 lbs. of sugar worth 28 lbs. of soap; how many lbs. of soap are 6 lbs. of tea worth?
47. A guinea is given to be divided among four persons, A, B, C, and D, with the direction that A is to have $\frac{1}{2}$; B, $\frac{1}{3}$; C, $\frac{1}{4}$; and D, $\frac{1}{8}$; but as this division is found to be impossible, find how the division must be made, so that each may receive his proper share.

* This is the same as if A contributed 5 times £280 for one month, and B, 6 times £266 13*s.* 4*d.* for one month; and in all questions of this kind, the sum contributed by each partner being multiplied by the number of months it was employed, the shares of the profit must be proportional to the resulting products. In like manner, the interest of a sum to be paid in 5 months is the interest of 5 times that sum to be paid in 1 month, and so on; so that if different sums are to be paid at different times, then if we multiply each sum by the number of months, or weeks, &c., which are to elapse before payment is to be made, and then divide the sum of the products by the sum of the bills, the quotient will be what is called the *equated time* in which all should be paid if they are paid a once. Examples 50 and 51 illustrate this principle.

48. What is the net weight of 152 cwt. 1 qr. 3 lbs., 10 lbs. per cwt. being allowed for *tare*, and $\frac{1}{26}$ of what remains for *tret*? See page 96.

49. If the rents of a parish amount to £2340 17*s*. 6*d*., and a rate of £137 10*s*. 8*d*. be levied, what portion of it must be paid by an estate of which the rental is £143 9*s*. 10*d*.?

50. A person engages to pay a debt of £2330 as follows: £630 in 5 months, £300 in 6 months, £500 in 7 months, and the remaining £900 in 8 months; but he afterwards proposes to pay the whole in one sum; in what time should it be paid? *

51. Goods are purchased on the following conditions, namely, that £50 are to be paid on the 1st of May, £64 on the 4th of June, £86 on the 1st of August, and £90 on the 5th of September; when should these sums be paid, if all are paid at once?

52. A ship's company take a prize of £4000, which is to be divided amongst them in proportion to their pay, and the time they have been on board. There are 6 officers, who have £6 a month each, and have been on board 6 months; 12 midshipmen, who have £2 a month each, and have been on board 4 months; 110 sailors, who have £1 10*s*. a month each, and have been on board 3 months; what share of the prize must each receive? †

* Agreeably to what is said in the foot-note, p. 177, we should multiply £630 by 5, £300 by 6, £500 by 7, and £900 by 8; but it is better to find how long *after* 5 months (the shortest of the periods) the whole payment should be made; as then the multipliers, instead of 5, 6, 7, and 8, will be only 1, 2, and 3, applied respectively to the sums *after* the first.

† This question may be worked by the same principle as that employed in the solution of the two preceding questions, combined with the principle of art. (112). The claim of the 6 officers at £6 a month, is the same as would be the claim of 6 × 6 officers at £1 a month; and the claim of these for 6 months' service is the same as would be the claim of 6 × 6 × 6 officers for 1 month's service. And the learner will observe, that the mode of solution recommended in this and in the two preceding questions is only a modification of the double rule of three, combined with the rule for proportional parts.

A TABLE

Of those Factors of the Composite Numbers from 75 to 10000, which fall within the Limits of the Multiplication Table.

No.	Factors.	No.	Factors.	No.	Factors.
75	5 5 3	405	9 9 5	1029	7 7 7 3
98	7 7 2	432	12 9 4	1056	12 11 8
105	7 5 3	441	9 7 7	1078	11 7 7 2
112	8 7 2	448	8 8 7	1089	11 11 9
125	5 5 5	462	11 7 6	1125	9 5 5 5
126	9 7 2	484	11 11 4	1134	9 9 7 2
128	8 8 2	486	9 9 6	1152	12 12 8
135	9 5 3	495	11 9 5	1155	11 7 5 3
147	7 7 3	504	9 8 7	1176	8 7 7 3
154	11 7 2	512	8 8 8	1188	12 11 9
162	9 9 2	525	7 5 5 3	1215	9 9 5 3
165	11 5 3	528	12 11 4	1225	7 7 5 5
168	8 7 3	539	11 7 7	1232	11 8 7 2
175	7 5 5	567	9 9 7	1296	12 12 9
176	11 8 2	576	12 12 4	1323	9 7 7 3
189	9 7 3	588	12 7 7	1331	11 11 11
192	12 8 2	594	11 9 6	1344	8 8 7 3
196	7 7 4	605	11 11 5	1372	7 7 7 4
198	11 9 2	616	11 8 7	1375	11 5 5 5
216	12 9 2	625	5 5 5 5	1386	11 9 7 2
224	8 7 4	648	9 9 8	1408	11 8 8 2
225	9 5 5	672	12 8 7	1452	12 11 11
231	11 7 3	675	9 5 5 3	1458	9 9 9 2
242	11 11 2	686	7 7 7 2	1485	11 9 5 3
243	9 9 3	693	11 9 7	1512	9 8 7 3
245	7 7 5	704	11 8 8	1536	8 8 8 3
252	12 7 3	726	11 11 6	1568	8 7 7 4
256	8 8 4	729	9 9 9	1575	9 7 5 5
264	11 6 4	735	7 7 5 3	1584	12 12 11
275	11 5 5	756	12 9 7	1617	11 7 7 3
288	12 12 2	768	12 8 8	1694	11 11 7 2
294	7 7 6	784	8 7 7 2	1701	9 9 7 3
297	11 9 3	792	12 11 6	1715	7 7 7 5
308	11 7 4	825	11 5 5 3	1728	12 12 12
315	9 7 5	847	11 11 7	1764	9 7 7 4
324	9 9 4	864	12 9 8	1782	11 9 9 2
336	12 7 4	875	7 5 5 5	1792	8 8 7 4
343	7 7 7	882	9 7 7 2	1815	11 11 5 3
352	11 8 4	891	11 9 9	1848	11 8 7 3
363	11 11 3	896	8 8 7 2	1875	5 5 5 5 3
375	5 5 5 3	924	12 11 7	1925	11 7 5 5
378	9 7 6	945	9 7 5 3	1936	11 11 4 4
384	8 8 6	968	11 11 8	1944	9 9 8 3
385	11 7 5	972	12 9 9	2016	9 8 7 4
392	8 7 7	1008	12 12 7	2025	9 9 5 5
396	11 9 4	1024	8 8 8 2	2048	8 8 8 4

No.	Factors.	No.	Factors.	No.	Factors.
2058	7 7 7 6	3872	11 11 8 4	6468	12 11 7 7
2079	11 9 7 3	3888	9 9 8 6	6534	11 11 9 6
2112	11 8 8 3	3969	9 9 7 7	6561	9 9 9 9
2156	11 7 7 4	3993	11 11 11 3	6615	9 7 7 5 3
2178	11 11 9 2	4032	9 8 8 7	6655	11 11 11 5
2187	9 9 9 3	4096	8 8 8 8	6776	11 [illegible] 8 7
2205	9 7 7 5	4116	12 7 7 7	6804	12 9 9 7
2268	9 9 7 4	4125	11 5 5 5 3	6860	10 7 7 7 2
2304	9 8 8 4	4158	11 9 7 6	6875	11 5 5 5 5
2352	8 7 7 6	4224	11 8 8 6	6912	12 9 8 8
2376	11 9 8 3	4235	11 11 7 5	7056	9 8 7 7 2
2401	7 7 7 7	4312	11 8 7 7	7128	11 9 9 8
2464	11 8 7 4	4356	11 11 9 4	7168	8 8 8 7 2
2475	11 9 5 5	4374	9 9 9 6	7203	7 7 7 7 3
2541	11 11 7 3	4375	7 5 5 5 5	7392	12 11 8 7
2592	9 9 8 4	4455	11 9 9 5	7425	11 9 5 5 3
2625	7 5 5 5 3	4536	9 9 8 7	7546	11 7 7 7 2
2646	9 7 7 6	4608	9 8 8 8	7560	12 10 9 7
2662	11 11 11 2	4704	12 8 7 7	7623	11 11 9 7
2673	11 9 9 3	4725	9 7 5 5 3	7744	11 11 8 8
2688	8 8 7 6	4752	11 9 8 6	7776	12 9 9 8
2695	11 7 7 5	4802	7 7 7 7 2	7875	9 7 5 5 5
2744	8 7 7 7	4851	11 9 7 7	7938	9 9 7 7 2
2772	11 9 7 4	4928	11 8 8 7	7986	11 11 11 6
2816	11 8 8 4	5082	11 11 7 6	8019	11 9 9 9
2835	9 9 7 5	5103	9 9 9 7	8064	9 8 8 7 2
2904	11 11 8 3	5145	7 7 7 5 3	8085	11 7 7 5 3
2916	9 9 9 4	5184	9 9 8 8	8192	8 8 8 4 4
3024	9 8 7 6	5292	12 9 7 7	8232	8 7 7 7 3
3025	11 11 5 5	5324	11 11 11 4	8316	12 11 9 7
3072	8 8 8 6	5346	11 9 9 6	8448	12 11 8 8
3087	9 7 7 7	5376	12 8 8 7	8505	9 9 7 5 3
3125	5 5 5 5 5	5445	11 11 9 5	8575	7 7 7 5 5
3136	8 8 7 7	5488	8 7 7 7 2	8624	11 8 7 7 2
3168	11 9 8 4	5544	11 9 8 7	8712	11 11 9 8
3234	11 7 7 6	5625	9 5 5 5 5	8748	12 9 9 9
3267	11 11 9 3	5632	11 8 8 8	9072	9 9 8 7 2
3375	9 5 5 5 3	5775	11 7 5 5 3	9075	11 11 5 5 3
3388	11 11 7 4	5808	11 11 8 6	9216	9 8 8 4 4
3402	9 9 7 6	5832	9 9 9 8	9261	9 7 7 7 3
3456	9 8 8 6	5929	11 11 7 7	9317	11 11 11 7
3465	11 9 7 5	6048	12 9 8 7	9375	5 5 5 5 5 3
3528	9 8 7 7	6075	9 9 5 5 3	9408	8 8 7 7 3
3564	11 9 9 4	6125	7 7 5 5 5	9504	12 11 9 8
3584	8 8 8 7	6144	12 8 8 8	9604	7 7 7 7 4
3645	9 9 9 5	6174	9 7 7 7 2	9625	11 7 5 5 5
3675	7 7 5 5 3	6237	11 9 9 7	9702	11 9 7 7 2
3696	11 8 7 6	6272	8 8 7 7 2	9801	11 11 9 9
3773	11 7 7 7	6336	11 9 8 8	9856	11 8 7 4 4

ANSWERS TO THE EXERCISES.

Numeration (pp. 4 and 5).

1. Two thousand, seven hundred and sixty-three. Thirty-five thousand, one hundred and sixty-two. Forty-five thousand, two hundred and eighty.
2. Fifty-six thousand, one hundred and six. Eighty-two thousand, and thirty. Nine hundred and ten thousand, two hundred and fifty-seven.
3. One hundred and seventy-three thousand, and four. Six million, seven hundred and eighty-nine thousand, five hundred and twenty-three. Three million, four hundred and eighty-six thousand, and twenty-five.
4. One million, one hundred and forty-two thousand, and sixty. One million, one hundred and ten thousand, one hundred and eleven. Four million, three hundred and sixty-two thousand, eight hundred.
5. Sixty-four million, three hundred and seventy thousand, two hundred and fifty-three. Ninety-nine million, eight hundred and seventy-four thousand, and sixty-two. Thirty-five million, six thousand, two hundred.
6. Seventy-three million, eight hundred and ninety-two thousand, five hundred and thirty-one. Eight hundred and seventy-five million, sixty-two thousand, and thirty-five. One hundred and seven million, nine hundred and twenty-six thousand, five hundred.
7. Seven thousand five hundred and thirty-nine million, three hundred and thirty-six thousand, two hundred and ten. Three hundred and twenty-six thousand nine hundred and seventy-two million, five hundred and seventy-three thousand, nine hundred and seventy-one. Four hundred and fifteen thousand, eight hundred and sixty-two million, three hundred and fourteen thousand, two hundred and three.
8. Seven hundred and thirty thousand two hundred and fifty-four million, sixty-two thousand, eight hundred and ten. One hundred and seventy-three thousand and four million, two hundred and two thousand, six hundred and four. Five hundred and two thousand one hundred and thirty million, sixty-five thousand, and eighty.

(Pages 4, 5.)

1. 1760.	5. 312500.	9. 1155240.
2. 546000.	6. 1177580.	10. 1314176.
3. 2000000.	7. 109915.	11. 6007944.
4. 621865.	8. 505107.	12. 20936468.

ADDITION (pp. 7, 8, 9).

1. 541. 2. 409. 3. 4641. 4. 12485.
5. 47115. 6. 797527. 7. 7581522. 8. 127297.
9. 11812. 10. In 1841, 18664761; in 1851, 20936468.
11. 518277. 12. 144 days; 6169016 visits. 13. 580347.
14. Newspapers, 593; advertisements, 2252550.

SUBTRACTION (pp. 12, 13, 14).

1. 471 and 1628. 2. 35891. 3. 10105 and 701208.
4. 103949. 5. 8888. 6. 1698069.
7. 82. 8. 1659330. 9. 965574.
10. 2271707. 11. 10892. 12. 877744.
13. 803899 pounds. 14. 9321416. 15. 203 pounds.
16. 2870784. 17. 21811085 pounds.
18. Born, 283001; died, 219052. 19. 10131.
20. *Nothing.* 21. 415539.

MULTIPLICATION (pp. 18, 19).

1. 1026. 2. 19044. 3. 35325. 4. 484344.
5. 7976563. 6. 3276. 7. 601604512.
8. 101904804408. 9. 7801178441640. 10. 49906.
11. 535914.* 12. 365000. 13. 269195.
14. 2250 and 7875. 15. 12375. 16. 86879544.

(Pages 24, 25.)

1. 114361. 2. 531786. 3. 2101120. 4. 471854788.
5. 478491860. 6. 2288575200. 7. 51947970000.
8. 16064348846. 9. 153787670242500.
10. 34141125200427. 11. 56604636.
12. 3024. 13. 759746144. 14. 94902500 miles.
15. 1238525174. 16. 427816 pounds. 17. 3486.
18. Gross earnings, 1177414 pounds; net earnings, 442157 pounds.

* The daily average stated in the question is taken from "The Companion to the Almanac for 1852;" but it is too great: the true daily average was 86379. See Ex. 11, p. 8.

DIVISION (pp. 29, 30).

1. $744\frac{4}{5}$.
2. $12013\frac{5}{6}$.
3. $15604\frac{6}{7}$.
4. $250797\frac{7}{8}$.
5. $5788233\frac{8}{9}$.
6. $1034602\frac{3}{11}$.
7. $899336\frac{7}{12}$.
8. $66459\frac{1}{4}$.
9. $124624\frac{3}{7}$.
10. 7048.
11. $3065170\frac{1}{8}$.
12. 86379.
13. 1628948.
14. 124373 pounds.
15. 1034280.
16. $155644\frac{1}{8}$ pounds.

(Pages 35, 36.)

1. $52\frac{19}{47}$.
2. $470\frac{62}{83}$.
3. $3500\frac{54}{342}$.
4. $14348\frac{473}{576}$.
5. $11222\frac{111}{843}$.
6. $14980\frac{1625}{2712}$.
7. $2602\frac{708}{3046}$.
8. $203\frac{1561}{402600}$.
9. $224\frac{14394}{78910}$.
10. $342\frac{61907}{80247}$.
11. $2272\frac{4167}{260833}$.
12. $314\frac{3021401}{3427500}$.
13. 1662 pounds.
14. 3846.
15. 13209.
16. 7727.
17. $78\frac{1}{2}$, *nearly*.
18. 62 times, *nearly*.

REDUCTION (pp. 47, 48).

1. 207809*d.*
2. 381907*f.*
3. 525600.
4. 662.
5. 35459.
6. 8550.
7. $83872\frac{1}{2}$.
8. 33880.
9. 8254.
10. 6624 qts.
11. 3411520 lbs.
12. 11474 lbs.
13. 432000 lbs.
14. 1254400 oz., 6272000*s.**
15. 288.
16. 38016.
17. 91476.
18. 1728000 qts.
19. 23rd June.
20. 81998.

(Pages 51, 52.)

1. £27 11*s.* $11\frac{1}{4}$*d.*
2. 225 m. 4 fur. 26 per. 1 yd.
3. 7 h. 57 m. 15 sec.
4. 2 lb. 2 oz. 16 dwt. 11 gr.
5. 78 t. 15 cwt. 1 qr. 4 lb.
6. 1539 yds. 1 qr. 3 na.
7. 266 ac. 2 roo. 17 po.
8. 21 sq. yds. 2 ft. 64 in.
9. 12500 gal.
10. 36° 48′ 50″.
11. 2 c. yds. 3 ft. 1504 in.
12. 44 t. 12 cwt. 3 qrs. 12 lb.
13. £650.
14. £30166 10*s.* 2*d.*
15. 13 d. $3\frac{15}{19}$ ho.
16. 39 d. $19\frac{15}{19}$ ho.
17. 200 t. 0 cwt. 2 qrs. 25 lbs.
18. £132546 1*s.* 6*d.*
19. 3 ho. 3 min. 20 sec.
20. 1961 t. 5 cwt. 20 lb.
21. 2603 c. yds. $10\frac{1}{8}$ ft.
22. 10 d. 4 h. 4 m. 4 s., the gal. *imp.*
23. 4240.
24. 10*s.*

* The reported weight was stated to be "about 35 tons;" but this estimate was too great. See the answer to Ex. 24, p. 107.

Addition of Compound Quantities (pp. 54, 55).

1. £38 11*s.* 7¾*d.*
2. £568 16*s.* 1¼*d.*
3. £1571 11*s.* 6½*d.*
4. 100 d. 7 h. 19 m.
5. 140 d. 19 h. 34 m.
6. 485 d. 3 h. 14 m. 29 s.
7. 69 lb. 3 oz. 9 dr.
8. 71 cwt. 1 qr. 22 lb.
9. 112 cwt. 1 qr. 26 lb. 13 oz.
10. 44 oz. 1 dwt. 5 gr.
11. 37 oz. 6 dwt.
12. 62 lb. 4 oz. 8 dwt. 16 gr.
13. 23 dr. 0 sc. 8 gr.
14. 38 oz. 0 dr. 1 sc.
15. 34 lb. 0 oz. 3 dr. 0 sc. 17 gr.
16. 548 yd. 1 ft. 2 in.
17. 23 fur. 25 po. ½ yd.
18. 425 m. 3 fur. 34 po. 4 yd.
19. 405 ac. 2 roo. 0 per. 12 yds.
20. 918 ac. 1 roo. 16 per. 9¼ yds.

Subtraction of Compound Quantities (pp. 56, 57, 58).

1. £15 14*s.* 8¼*d.*
2. £206 12*s.* 8¼*d.*
3. £2283 10*s.* 8¼*d.*
4. 6 d. 19 h. 58 m.
5. 67 d. 21 h. 46 m.
6. 16 h. 32 m. 32 s.
7. 74 yd. 1 ft. 5 in.
8. 188 yd. 1 ft. 9 in.
9. 12 per. 5 yds. 0 ft. 2 in.
10. 5° 33′ 54″.
11. 8° 36′ 49″.
12. 23° 54′ 47″.
13. 3 d. 22 h. 43 m. 34 sec.
14. 3 t. 16 cwt. 1 qr. 20 lb.
15. 6 t. 1 cwt. 1 qr. 13 lbs.
16. 1 t. 3 cwt. 2 qrs. 3 lb.
17. 75 m. 0 fur. 15 per. 2½ yds.
18. 15 m. 2 fur. 34 per. 2½ yds.
19. 101 m. 1 fur. 27 per. 4½ yds.
20. 53 ac. 2 roo. 22 per. 25¼ yds.
21. 6 ac. 0 roo. 29 per. 29½ yds.
22. 2 roo. 33 per. 15¼ yds.
23. 10 oz. 18 dwt. 6 gr.
24. 1 lb. 5 oz. 14 dwts. 16 gr.
25. 5 lb. 0 oz. 7 dwts. 13 gr.
26. 38 c. yds. 26 ft. 1697 in.
27. 63 c. yds. 20 ft. 1591 in.
28. 315 c. yds. 4 ft. 1547 in.
29. 85 sq. yds. 3 ft. 32 in.
30. 201 sq. yds. 8 ft. 122 in.
31. 306 sq. yds. 4 ft. 83 in.
32. 8 gal. 1 qt. 1 pt.
33. 1708 gal. 1 qt.
34. 684 gal. 2 qts. 1 pt.
35. 1 bu. 0 pk. 1 gal.
36. 5 bu. 0 pk. 0 gal. 2 qt.
37. 15 bu. 1 pk. 0 gal. 3 qt.

Multiplication of Compound Quantities (pp. 60, 61).

1. £162 2*s.* 8½*d.*
2. £348 11*s.* 2*d.*
3. £1507 16*s.* 3*d.*
4. £2397 6*s.* 5¾*d.*
5. £6614 16*s.* 9¾*d.*
6. £550 16*s.* 3*d.*
7. £7372 7*s.* 1½*d.*
8. £16560 1*s.* 6*d.*

9. £92293 13*s*. $6\frac{3}{4}$*d*. 10. £614350 10*s*. $3\frac{3}{4}$*d*.
11. 1153 m. 6 fur. 4 per. 3 yds.
12. 1760 m. 7 fur. 29 per. 3 yds.
13. 5945 m. 0 fur. 29 per. $\frac{1}{2}$ yd. 14. 496 m. 5 fur. 2 per.
15. 1019 m. 3 fur. 5 per. 16. 26677 m. 4 fur. 10 per. 5 yd.
17. 8631 d. 0 h. 48 m. 45 s. 18. $3907° 45' 20\frac{1}{4}''$.
19. 3567 t. 4 cwt. 1 qr. 13 lb. 20. 7080 a. 2 roo. 7 po.
21. 1679 sq. yds. 7 ft. 129 in. 22. 6095 oz. 14 dwt. 19 gr.

Division of Compound Quantities (pp. 64, 65, 66).*

1. £18 12*s*. $0\frac{1}{2}$*d*. 2. £16 19*s*. $6\frac{1}{4}$*d*.
3. £16 1*s*. 8*d*. 4. £1 9*s*. $8\frac{1}{2}$*d*.
5. £4 12*s*. 11*d*. 6. £3 1*s*. $4\frac{1}{4}$*d*.
7. 3 qrs. 5 lb. 2 oz. 8. 74 m. 7 fur. 9 po. 1 yd.
9. $1° 1' 48''$. 10. 2 d. 19 h. 39 m. 26 s.
11. 16 ac. 3 roo. 39 per. 27 yds. 8 ft.

In the following answers the complete quotients, including the fractional part in each case, are inserted.

12. $5\frac{596}{641}$. 13. $12\frac{3426}{6881}$. 14. $3\frac{20763}{33511}$.
15. $6\frac{36309}{45447}$. 16. $13\frac{447}{602}$. 17. $15\frac{4519}{7262}$.
18. $6\frac{2518}{47118}$. 19. $58\frac{83}{101}$. 20. £46 14*s*. 6*d*.
21. £5417 2*s*. $4\frac{3}{4}$*d*. + $\frac{275}{607}$*f*. 22. 4*d*.
23. Total, £93 0*s*. $3\frac{3}{4}$*d*.; share, £23 5*s*. $0\frac{3}{4}$*d*. + $\frac{3}{4}$*f*.
24. 16 dwt. $13\frac{8}{12}$ gr. 25. $1327433\frac{1}{2}$, and $14\frac{1}{2}$ gr. over.
26. £1216796 17*s*. 6*d*. 27. 4*s*. $6\frac{1}{4}$*d*. + $\frac{70}{106}$*f*.
28. Weight, 8541 lb. 8 oz. troy; height, 10 times the Monument, and 63 ft. 4 in. more.
29. £1869 12*s*. $9\frac{1}{2}$*d*. 30. 108 lb. 6 oz. 1 dwt. 16 gr.
31. Weight of sovereigns, 51094 lb. 10 oz. 3 dwt. 21 gr.; weight of pure gold, 46845 lb. 6 oz. 19 dwt. 17 gr.

Reduction of Fractions (p. 70).

1. $\frac{23}{8}$. 2. $\frac{122}{7}$. 3. $11\frac{7}{12}$. 4. $122\frac{1}{17}$.
5. $\frac{6445}{27}$. 6. $259\frac{15}{123}$. 7. $\frac{88381}{19}$. 8. $40\frac{51}{317}$.

* In these answers fractions of a farthing are neglected; and in general, fractions of the lowest denomination, inserted in the answer, are suppressed. This is done, because the fractions up to Ex. 12 are too insignificant to deserve notice. They are given however in the Key.

Common Denominator (pp. 72, 74).

1. $\frac{15}{30}, \frac{20}{30}, \frac{6}{30}$.
2. $\frac{45}{60}, \frac{24}{60}, \frac{20}{60}$.
3. $\frac{189}{315}, \frac{90}{315}, \frac{140}{315}$.
4. $\frac{54}{144}, \frac{120}{144}, \frac{336}{144}$.
5. $\frac{539}{693}, \frac{189}{693}, \frac{495}{693}$.
6. $\frac{240}{1560}, \frac{1404}{1560}, \frac{1430}{1560}$.
7. $\frac{840}{1120}, \frac{448}{1120}, \frac{960}{1120}, \frac{140}{1120}$.
8. $\frac{384}{480}, \frac{180}{480}, \frac{400}{480}, \frac{240}{480}$.
9. $\frac{360}{420}, \frac{252}{420}, \frac{560}{420}, \frac{315}{420}$.
10. $\frac{1540}{2772}, \frac{756}{2772}, \frac{792}{2772}, \frac{693}{2772}$.
11. $\frac{126}{252}, \frac{108}{252}, \frac{882}{252}, \frac{140}{252}$.
12. $\frac{945}{1260}, \frac{756}{1260}, \frac{180}{1260}, \frac{700}{1260}$.
13. $\frac{6}{8}, \frac{5}{8}, \frac{2}{8}$.
14. $\frac{30}{45}, \frac{27}{45}, \frac{35}{45}$.
15. $\frac{120}{168}, \frac{224}{168}, \frac{91}{168}$.
16. $\frac{3}{9}, \frac{3}{9}, \frac{1}{9}$.
17. $\frac{5}{108}, \frac{24}{108}, \frac{22}{108}$.
18. $\frac{35}{25}, \frac{4}{25}, \frac{2}{25}$.
19. $\frac{4}{18}, \frac{10}{18}, \frac{7}{18}, \frac{9}{18}$.
20. $\frac{30}{60}, \frac{20}{60}, \frac{15}{60}, \frac{12}{60}$.
21. $\frac{15}{60}, \frac{12}{60}, \frac{10}{60}, \frac{18}{60}$.
22. $\frac{8}{36}, \frac{12}{36}, \frac{15}{36}, \frac{126}{36}$.
23. $2\frac{2}{12}, \frac{5}{12}, \frac{8}{12}, \frac{3}{12}$.
24. $3\frac{15}{75}, \frac{50}{75}, \frac{9}{75}, \frac{20}{75}$.

Addition of Fractions (p. 76).

1. $1\frac{11}{140}$.
2. $1\frac{41}{120}$.
3. $1\frac{361}{420}$.
4. $6\frac{61}{72}$.
5. $6\frac{17}{18}$.
6. $14\frac{19}{24}$.
7. $8\frac{61}{72}$.
8. $1\frac{163}{270}$.
9. $4\frac{317}{400}$.
10. $5\frac{379}{1260}$.
11. $3\frac{851}{1080}$.
12. $7\frac{38}{189}$.
13. $1\frac{2773}{3190}$.
14. $8\frac{25279}{41580}$.
15. $5\frac{271}{756}$.
16. $7\frac{1636}{5005}$.

Subtraction of Fractions (pp. 76, 77).

1. $1\frac{1}{20}$.
2. $1\frac{13}{56}$.
3. $\frac{41}{84}$.
4. $\frac{5}{16}$.
5. $\frac{34}{35}$.
6. $\frac{87}{110}$.
7. $\frac{43}{72}$.
8. $\frac{1}{80}$.
9. $\frac{71}{96}$.
10. $2\frac{59}{220}$.
11. $2\frac{2}{99}$.
12. $2\frac{5}{12}$.
13. $2\frac{51}{80}$.
14. $\frac{11}{105}$.
15. $1\frac{121}{420}$.
16. $-\frac{6}{55}$.*
17. $6\frac{19}{126}$.
18. $1\frac{17}{45}$.
19. $\frac{167}{240}$.
20. $4\frac{262}{495}$.
21. $10\frac{29}{45}$.
22. $\frac{25}{48}$.
23. $\frac{239}{264}$.
24. $2\frac{173}{180}$.

Multiplication of Fractions (p. 80).

1. $\frac{7}{90}$.
2. $10\frac{10}{77}$.
3. $\frac{7}{80}$.
4. $\frac{10}{77}$.
5. $3\frac{71}{108}$.
6. $12\frac{6}{7}$.
7. $120\frac{11}{24}$.
8. $347\frac{11}{45}$.
9. $14\frac{16}{35}$.
10. $1\frac{19}{20}$.
11. $10\frac{65}{81}$.
12. $2\frac{487}{504}$.
13. $84\frac{21}{40}$.
14. $204\frac{1}{6}$.
15. $4\frac{4}{7}$.
16. $\frac{314}{675}$.

Division of Fractions (p. 81).

1. $1\frac{1}{11}$.
2. $1\frac{23}{40}$.
3. $\frac{164}{273}$.
4. $\frac{381}{749}$.
5. $\frac{1}{6}$.
6. $50\frac{5}{8}$.
7. $\frac{48}{175}$.
8. $\frac{143}{540}$.
9. $\frac{60}{77}$.
10. $2\frac{43}{81}$.
11. 198.
12. $\frac{110}{171}$.

* This means that the subtractive quantities *exceed* the additive quantities by $\frac{6}{55}$.

(Page 83.)

1. $\frac{11}{30}$. 2. $\frac{17}{60}$. 3. $\frac{47}{252}$. 4. $\frac{3}{70}$.
5. $\frac{29}{36}$. 6. $\frac{13}{80}$. 7. $\frac{3}{22}$. 8. $\frac{115}{378}$.
9. $\frac{259}{1000}$. 10. $\frac{1487}{10800}$. 11. $\frac{65}{168}$. 12. $\frac{7}{9}$.
13. $\frac{49}{150}$. 14. $\frac{289}{818}$. 15. $\frac{17073}{223366}$.

Multiplication and Division of Concrete Quantities (p. 85).

1. £160 6*s.* $7\frac{1}{2}$*d.*
2. 19*s.* $5\frac{3}{4}$*d.* + $\frac{5}{77}$*f.*
3. £658 0*s.* $11\frac{1}{4}$*d.* + $\frac{3}{4}$*f.*
4. 7 miles, $2\frac{34}{85}$ fur. (It should have been *rides.*)
5. 13*s.* $4\frac{1}{2}$*d.* + $\frac{6}{7}$*f.*
6. A, £178 11*s.* $5\frac{1}{7}$*d.*; B, £138 17*s.* $9\frac{1}{3}$*d.*; C, £307 10*s.* $9\frac{11}{21}$*d.*
7. £1549096 17*s.* 6*d.*
8. £375.
9. $\frac{5765}{15561}$.
10. 49 d. 4 h. 0 m. $49\frac{5}{18}$ sec.

Greatest Common Measure (pp. 88, 89).

1. 19. 2. 17. 3. 3. 4. 17.
5. 97. 6. It is already in its lowest terms.
7. $\frac{103}{186}$. 8. $\frac{247}{547}$.
9. No common measure. 10. 13 rods.
11. Price, per acre, £19: number of acres bought by A, 17; by B, 24; and by C, 29. 12. 80.

Least Common Multiple (p. 91).

1. 360. 2. 81. 3. 2520. 4. 1200.
5. 7560. 6. 10540068. 7. 232792560.

Practice (pp. 96, 97).

1. £104 10*s.* 6*d.*
2. £159 4*s.* 4*d.*
3. £440 15*s.* 1*d.*
4. £512 13*s.* $11\frac{1}{2}$*d.*
5. £224 3*s.* $2\frac{1}{2}$*d.*
6. £733 0*s.* $1\frac{1}{4}$*d.*
7. £541 18*s.* $9\frac{3}{4}$*d.*
8. £1328 1*s.* $11\frac{1}{2}$*d.*
9. £2794 19*s.* $0\frac{1}{4}$*d.*
10. £1442 19*s.* $10\frac{1}{2}$*d.*
11. £2353 2*s.* 1*d.*
12. £4149 5*s.* $11\frac{1}{4}$*d.*
13. £3 14*s.* $2\frac{3}{4}$*d.*
14. £3 19*s.* 10*d.*
15. £84 0*s.* $9\frac{1}{4}$*d.*
16. £35 13*s.* $3\frac{3}{4}$*d.*
17. £1 7*s.* $11\frac{1}{2}$*d.*
18. £23 11*s.* 3*d.*
19. £18 9*s.* 5*d.*
20. £426 12*s.* 1*d.*
21. £4 13*s.* $2\frac{1}{4}$*d.*
22. £4989 0*s.* $10\frac{1}{4}$*d.*

23. £176 16*s*. 3*d*.
24. £424800 19*s*.
25. £1351952 13*s*.
26. £54 12*s*. (stone=8 lb.)
27. £205 19*s*. 6*d*.
28. £76 11*s*. 3*d*.
29. Reduction per cwt., £2 3*s*.; per stone, 3*s*. $0\frac{3}{4}$*d*.
30. £12763133 16*s*. 11*d*.

Rule of Three, or Simple Proportion (pp. 106, 107).

1. 15*s*. $8\frac{1}{17}$*d*.
2. £23 19*s*. $1\frac{1}{4}$*d*. + $\frac{1}{3}$*f*.
3. £7 11*s*. $7\frac{3}{4}$*d*. + $\frac{1}{2}$*f*.
4. 48 persons.
5. 400.
6. $6\frac{2}{3}$ oz.
7. £5 11*s*. $6\frac{3}{4}$*d*. + $\frac{3}{7}$*f*.*
8. £9 18*s*.
9. £23 10*s*. $9\frac{3}{4}$*d*.
10. 270.
11. £4240.
12. 129478.
13. $46\frac{29}{40}$.
14. 14 oz. 11 dwt. 16 gr. *troy*.
15. 10 lb. *avoir*.
16. 5 min. $27\frac{3}{11}$ sec. past 7 o'clock.
17. 27 min. $16\frac{4}{11}$ sec. past 5.
18. $72\frac{6}{11}$.
19. £31 5*s*. $7\frac{3}{4}$*d*.
20. 2 lb. $8\frac{24}{257}$ oz.
21. $14\frac{8}{11}$ miles.
22. £767.
23. 3 oz. 7 dwt. $6\frac{6}{11}$ gr.
24. 30 tons 12 cwt. $27\frac{3}{7}$ lb.

Double Rule of Three, or Compound Proportion (pp. 111, 112).

1. 20 horses.
2. $17\frac{7}{25}$ days.
3. £1 4*s*.
4. 125 reams.
5. $43\frac{1}{6}$ days.
6. 56 days.
7. 3 lb.
8. £72.
9. £39 5*s*. $1\frac{1}{2}$*d*.
10. £16 10*s*.

Reduction of Fractions to Decimals (p. 117).

1. ·4375.
2. ·078125.
3. ·536.
4. ·372.
5. ·0064.
6. ·00448.
7. ·5375.
8. ·005541.
9. ·6857143.
10. ·51282.
11. ·733333.
12. ·9.

Addition and Subtraction of Decimals (p. 118).

1. 279·385.
2. 6343·4214.
3. 368·82.
4. 105·5928
5. 1·2949.
6. 296·6108.

* Although fractions of a farthing are introduced into some of these answers, the learner need not take any account of them.

Multiplication of Decimals (p. 119).

1. 209·226285.	2. 134·70368.	3. 114192.
4. ·002736.	5. ·000000777.	6. ·0552762.
7. ·00058128.	8. 13·44.	9. ·4578384384.
10. ·0393696.	11. 574·056.	12. ·0000896.
13. 195·2533006056.	14. 21·8630035575.	

Contracted Multiplication (p. 125).

1. 1308·0037.	2. 482·554.	3. 54·4442.*
4. 105·928788.	5. 350	6. 590·324.
7. 1·686659.	8. ·002517.	9. 235·104.
10. ·34028		

Division of Decimals (p. 129).

1. 213·728.	2. 168·448.	3. 320·988.
4. ·066.	5. 7·845.	6. 34097·027.

Contracted Division (p. 132).

1. 6·598.	2. 4343·91.	3. ·00429.
4. 7·845.	5. 14·31.	6. 1·1715.
7. 5·348.	8. 1624·7.	9. ·865256.
10. ·000156.	11. 71,00000.†	12. ·0403.
13. 196·8.	14. ·09547766.	15. 1·684479.
16. 1·196.	17. ·2645653.	18. 18·28.
19. 93·54286.	20. ·098135.	21. 5·348306.
22. ·0981354.	23. ·0042913, and ·040266.	
24. 52·486.		

* If this product had been computed to *five* places, and then the fifth decimal rejected, the result would have been 54·4444, which is the more accurate one. In the answers which follow, the extra decimal will always be computed for, and then rejected; as recommended at p. 126.

† The quotient here is 7 million 1 hundred thousand,—this 1 being true to the nearest unit; that is to say, it is nearer the truth than a 0 or a 2 would be in its place. Of the following figures we can infer nothing, since the final decimal 4 of the divisor is confessedly affected with error. The quotient therefore can only be approximated to as far as the two leading figures of it are concerned. The decimals in the curtailed divisor must be extended before the figures, replaced above by 0's, can be found.

Application of Decimals to Concrete Quantities (pp. 134, 135).

1. 15 poles.
2. 3° 37′ $42\frac{3}{5}$″.
3. 9 cwt. 1 qr.
4. $5\frac{1}{4}d$.
5. £1 16*s*. $6\frac{3}{4}d$.
6. 5 oz. 12 dwt. 15·744 gr.
7. £19·86354.
8. ·0020307.
9. ·0375.
10. ·2422419.
11. 10 oz. 0 dwt. $0\frac{3}{5}$ gr.
12. £4 14*s*. $8\frac{3}{4}d. + \frac{1}{4}f$.
13. 7*s*. $2\frac{1}{2}d. + \frac{2}{5}f$.
14. $\frac{24}{25}$, and $\frac{817}{5000}$.
15. 12 h. 44 m. 2·86368 sec.
16. 7912·3 miles.
17. 27747‸00 miles, the figures that would replace the 0's are not to be depended on.
18. £1.
19. 19*s*. $4\frac{1}{5}d$.
20. £5 16*s*. 3*d*.
21. 31 ac. 2 roo. 14·736 per.
22. £47 6*s*. 4·154*d*.
23. 4*s*. 8·3108*d*.
24. ·022191.

Recurring Decimals (p. 138).

$\cdot\dot{1}3\dot{5} = \frac{5}{37}$. $2\cdot4\dot{1}\dot{8} = 2\frac{23}{55}$. $\cdot5\dot{9}2\dot{5} = \frac{592}{999}$.

$\cdot004\dot{4}\dot{9} = \frac{89}{19800}$. $3\cdot\dot{7}56\dot{9} = 3\frac{841}{1111}$. $621\cdot\dot{6}2\dot{1} = 621\frac{23}{37}$.

$\cdot\dot{0}243\dot{9} = \frac{1}{41}$. $\cdot\dot{8}5714\dot{2} = \frac{6}{7}$. $1\cdot\dot{0}37\dot{8} = 1\frac{42}{1111}$.

$\cdot008\dot{4}9713\dot{3} = \frac{83}{9768}$.

Extraction of the Square Root (p. 148).

1. 5·637566.
2. 10·7376607.
3. ·5688645.
4. 3·316625.
5. 687·936.
6. 1·73205081.
7. 950625.
8. 19·10497317.
9. 5·692.
10. ·027.
11. 28·0067.
12. 8·8985.
13. 8·290717701.
14. $5\frac{4}{9}$.
15. $10\frac{1}{6}$.
16. 4·16833.
17. 3·952847.
18. 28·18155425.
19. 5·9049.
20. 2·7664333.

Extraction of the Cube Root (p. 154).

1. 97.
2. 375.
3. 276.
4. 4321.
5. 7435.
6. 19·86228.
7. 4·867132.
8. 93·7.
9. 42·7839.
10. 23·11204.
11. 4·9793386.
12. 2·080083823.

INTEREST, &c. (pp. 162, 163).

1. £245 13*s.* 4*d.*	2. £72 16*s.*	3. £3 8*s.* $0\frac{1}{2}$*d.*
4. £16 4*s.* $10\frac{1}{4}$*d.*	5. £3 1*s.* $2\frac{3}{4}$*d.*	6. £68 15*s.* $9\frac{1}{2}$*d.*
7. £123 5*s.* 6*d.*	8. £6825.	9. 4 years.
10. $4\frac{1}{2}$.	11. £23 1*s.* $8\frac{3}{4}$*d.*	12. 5·236 nearly.
13. 12·17.	14. $20\frac{3}{10}$ nearly.	

DISCOUNT (pp. 165, 166).

1. £1251 11*s.* 1*d.**	2. £216 15*s.* 9*d.*
3. £554 12*s.* 4*d.*	4. £1548 19*s.* 9*d.*
5. £38 18*s.* 6*d.*	6. £149 0*s.* 7*d.*
7. £9 16*s.*	

BROKERAGE, COMMISSION, INSURANCE, &c. (p. 169).

1. £129 9*s.*	2. £499 10*s.*
3. £2000.	4. £2056 17*s.* 11*d.*
5. £32 17*s.* $2\frac{1}{2}$*d.*	6. 16*s.* 9*d.*
7. £1165 19*s.* 6*d.*	8. £14 11*s.* $10\frac{1}{2}$*d.*
9. £6384 12*s.* 3*d.*	10. £28 16*s.* 3*d.*

PROPORTIONAL PARTS (pp. 171, 172).

1. A's share, £150; B's, £195; C's, £210.
2. Nitre, $85\frac{3}{25}$ lb.; charcoal, $15\frac{17}{25}$ lb.; sulphur, $11\frac{1}{5}$ lb.
3. Pure gold, 4 dwt. $22\frac{58}{89}$ gr.; alloy, $10\frac{70}{89}$ gr.
4. Pure silver, 4 oz. $12\frac{1}{2}$ dwt.; alloy, $7\frac{1}{2}$ dwt.
5. Oxygen, 889 oz. hydrogen, 111 oz.
6. A, £72; B, £48; C, £45.
7. Tin, 16 cwt. 3 qr. $10\frac{6}{19}$ lb.; lead, 2 cwt. 1 qr. $0\frac{12}{19}$ lb.; brass, 3 qr. $17\frac{1}{19}$ lb.
8. A, £95 1*s.* $8\frac{3}{4}d.+\frac{101}{153}f.$; B, £71 6*s.* $3\frac{1}{2}d.+\frac{114}{153}f.$; C, £57 1*s.* $0\frac{1}{2}d.+\frac{30}{153}f.$; D, £47 10*s.* $10\frac{1}{4}d.+\frac{127}{153}f.$; E, £40 15*s.* $0\frac{1}{4}d.+\frac{87}{153}f.$

CHAIN RULE (p. 172).

1. 64 lb.	2. $72\frac{24}{65}$ lb.	3. $5750\frac{280}{493}$.

* The results are computed to the nearest penny, which is in accordance with commercial practice.

DUODECIMALS (p. 174).

1. 270 sq. ft. 2 pts. 3 sq. in.
2. 363 sq. ft. 10 pts. 6 sq. in.
3. 1942 sq. ft. 1 pt.
4. 1612 sq. ft. 10 pts. 6 sq. in.
5. 1331 sq. ft. 11 pts. 3 sq. in.
6. 854 sq. ft. 7 pts.
7. 233 sq. ft. 6 sq. in.
8. 3117 sq. ft. 10 pts. 4 sq. in.

MISCELLANEOUS EXAMPLES (p. 175).

1. £·02770449 and $\frac{5}{8}$*s.*
2. $\frac{5}{6}$.
3. £37 5*s.* 3*d.*
4. £35 12*s.* 3$\frac{1}{2}$*d.*
5. £253 10*s.* 0$\frac{3}{4}$*d.*
6. £2 18*s.* 9*d.*
7. 30 men.
8. £3135 15*s.* 6$\frac{1}{4}$*d.*
9. 13*s.* 3*d.*
10. £5829 17*s.* 5*d.*
11. 8*s.* 8*d.*
12. £5 10*s.* 4$\frac{3}{4}$*d.*
13. £350 5*s.*
14. 9 min. 23$\frac{1}{5}$ sec.
15. 6 ho. 15 min.
16. 2184.
17. 10 min. 54$\frac{6}{11}$ sec. to 10 o'clock.
18. 173 and 611.
19. 42·784 and ·6918984.
20. 16$\frac{2}{3}$ days.
21. 36 lbs.
22. 389017, 531441, and 275.
23. 17*s.* 6*d.*
24. £8978 13*s.* 6*d.*
25. ·02734375 lb.
26. £27 3*s.* 0$\frac{1}{4}$*d.*
27. £3623 3*s.* 9*d.*
28. £2 2*s.* 5*d.*
29. £5.
30. £1 11*s.* 0$\frac{1}{2}$*d.*
31. 28 cwt. 1 qr. 9$\frac{1}{3}$ lbs.
32. Brandy, 18*s.*; Rum, 14*s.*; Geneva, 10*s.* 6*d.*
33. £344 11*s.* 6$\frac{3}{4}$*d.*
34. 2$\frac{3}{28}$ and $\frac{2}{15}$.
35. ·793701 and 1·930979.
36. A's share, £23 13*s.* 6*d.*; B's, £35 4*s.* 2*d.*; C's, £41 2*s.* 4*d.*
37. 5$\frac{15}{28}$ days.
38. 6001$\frac{53}{72}$ yds.
39. £38 8*s.* 5$\frac{3}{4}$*d.*
40. A's share, £154 15*s.* 2*d.*; B's, £176 17*s.* 4*d.*
41. A's share, £25 10*s.*; B's, £37 2*s.*; C's, £24 4*s.*
42. ·038925 and 17·83.
43. Copper, 112$\frac{1}{2}$ lbs.; tin, 37$\frac{1}{2}$ lbs.
44. Pure spirit, 40$\frac{8}{25}$ gallons; water, 43$\frac{17}{25}$ gallons.
45. 5 per cent.
46. 96$\frac{32}{65}$ lbs.
47. A's share, 8*s.* 2$\frac{2}{11}$*d.*; B's, 5*s.* 5$\frac{5}{11}$*d.*; C's, 4*s.* 1$\frac{1}{11}$*d.*; D's, 3*s.* 3$\frac{3}{11}$*d.*
48. 133 cwt. 1 qr. 12 lb.
49. £8 8*s.* 7$\frac{1}{4}$*d.*
50. 6 m. 2 w. 6 d.
51. July 14th.
52. Each officer, £178 8*s.* 9$\frac{1}{4}$*d.*; midshipman, £39 13*s.* 0$\frac{3}{4}$*d.*; sailor, £22 6*s.* 1*d.*

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